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# COLOUR GRADIENTS IN CLUSTERS OF GALAXIES ( I )

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*Abstract.* The analysis of the photometrical observations for galaxies with $z > 0.2$ is of great importance for both Extragalactic Astronomy and Cosmology. This reveals a large variety of phenomena and cosmological effects. We use the method of *colour gradients* in clusters of galaxies in order to study some of these evolutive effects. The results concerning the colour gradients in 15 clusters of galaxies with $0 < z \leq 0.81$ and the implications in the morphology and evolution of the clusters are presented. The connexion between the colour gradient and the Butcher-Oemler effect, the problem of colour evolution of galaxies in clusters are also discussed.

*Key words:* cosmology – clusters evolution – colour gradients.

## 1. INTRODUCTION

One of the most important advantages of observational cosmology is the possibility of obtaining informations about the evolutionary history of galaxies, using the observations of distant clusters of galaxies.

The multicolour and spectroscopic data on distant clusters of galaxies combined with the morphological informations on such clusters, obtained by Hubble Space Telescope (HST), represent a powerful combination of data necessary for:

– investigation of the conditions of galaxy formation, evolution of stellar populations, star formation history;

– understanding of the processes considered to be responsible for galaxies evolution (these processes are various and complex, especially in the case of clusters of galaxies).

Generally, the effect of evolution contains two components:

A) The evolution of the galaxies as individual objects (the intrinsic evolution of galaxies) – Bruzual 1981–1985; Rocca Volmerange, Guiderdoni 1985–1989 spectrophotometrical models with evolution;

B) The evolution of systems of galaxies (clusters, superclusters ..).

The second component contains at least two subcomponents:

B.1.) The physical and chemical evolution of galaxies in clusters (the dependence on cluster environment), studied by us by means of the colour gradients (Suran and Popescu 1988–1989, Suran et al. 1991);

B.2.) Possible morphological modification (in time) of cluster population, along the Hubble sequence.

From their photometry of two distant clusters of galaxies at $z = 0.4$ (3C295 – Minkovski 1960; Cl0024+1654 – Zwicky 1959), Butcher and Oemler (1978) reported

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that these two distant rich concentrated clusters have a significant population of blue extended objects, in comparison to that seen in present-day clusters of similarly high richness and central concentration. Usually, such blue galaxies are spiral galaxies at the present epoch, but in the nearly rich concentrated clusters, spiral galaxies represent a small fraction of the population.

Almost all subsequent photometric studies of rich galaxy clusters out to $z = 0.8$ have indicated a significant evolution of the ***blue galaxy content*** in the cores of these clusters (an increase in the fraction of blue galaxies at higher redshifts, especially at redshifts larger than 0.2).

HST images show that these blue galaxies are low luminosity spiral galaxies "Sp" (Dressler et al 1994, Couch et al 1994). Present-day rich concentrated clusters of galaxies are dominated by elliptical galaxies (E) and lenticular (S0) galaxies (Dressler 1980), most of them being low luminosity dwarfs.

Based on spectroscopic studies (Dressler and Gunn 1982; Couch and Sharples 1987), an alternative view of this effect corresponds to an increase in the fraction of ***active galaxies*** such as: emission-line galaxies (normal spirals or starbursts), AGN (active galactic nuclei), E+A galaxies (post-starburst galaxies, which are characterized by strong Balmer absorption lines and weak or no emission line + spectra like the old stellar populations of E galaxies), which present signs of recent star formation and/or nuclear activity.

There were some questions with uncertain answers concerning:

– the problem whether the fraction of blue galaxies, observed in distant clusters, represents the presence of foreground clusters or is due to the fluctuations in the field galaxy population;

– the nature of these blue galaxies which are members of the cluster – these galaxies can proceed from conversion of spiral galaxies (S) to lenticular S0 galaxies by processes as: mergers (Icke 1985), interactions, gas compression in the cluster environment with high pressure (Dressler and Gunn 1983), tidal field of the cluster (Byrd and Valtonen 1990).

Dressler et al. (1994), analyzing their HST images of the cluster 0939 + 4713 ($z = 0.41$), observed an important number of spirals which present signs of tidal interactions and/or mergers. The tides and ram pressure stripping are acting efficiently only near the center of the cluster. The authors conclude that, in distant clusters, strong dynamical interactions between galaxies may be responsible for the strong starbursts observed in many blue cluster members, if these interactions result in mergers.

So, each of the mentioned processes can produce starbursts, but none can identify the remnants of the distorted blue galaxies in the present-day clusters and explain the morphological evolution.

In 1984 Butcher and Oemler claimed the dependence of the blue galaxies fraction on $z$:

$$f_b = f_b(z),$$

using a sample of clusters with $0 < z \leqslant 0.54$.

Suran and Popescu 1988–1989 used the concept of colour gradients in clusters of galaxies:

$$a_{B-V} = \mathrm{d}(B-V)/\mathrm{d}R$$

as a measure of the clusters evolution.

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The dependence of these gradients on $z$:

$$a_{B-V} = a_{B-V}(z)$$

was estimated to have almost the same behaviour with Butcher-Oemler effect (BO effect) for clusters with $0 < z < 0.5$. The colour gradients can be a possible explanation of this effect.

In 1992 Molinari et al. suspected the radial dependence of BO effect in clusters:

$$f_B = f_B(R).$$

This thing represents an indirect confirmation of the connection between colour gradients in clusters of galaxies and BO effect.

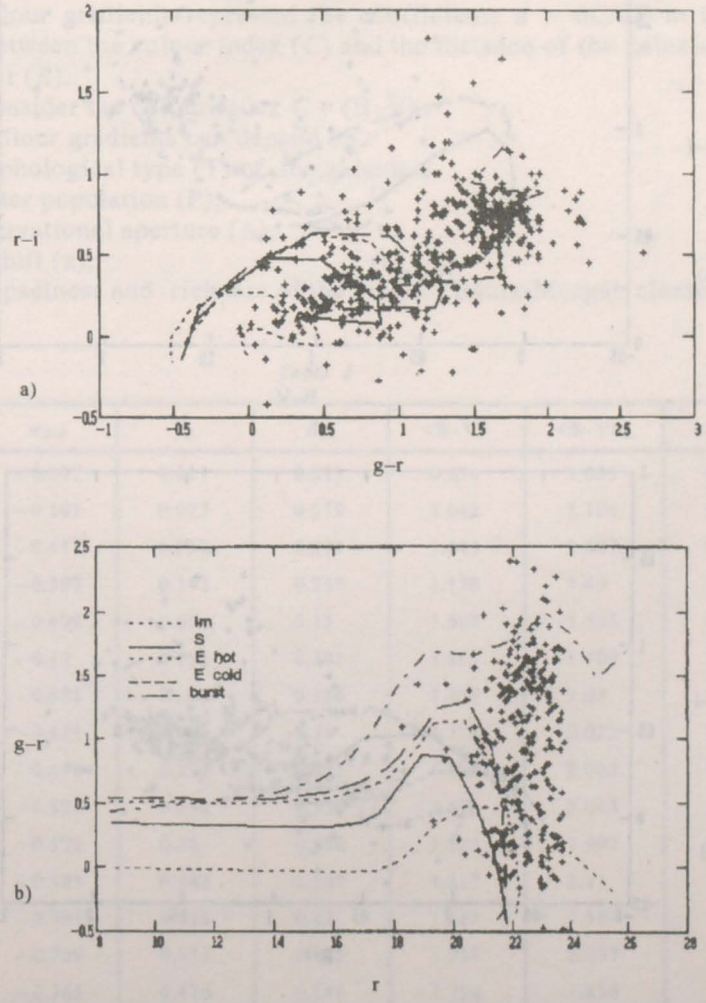



Fig. 1 a, b – The colour-colour and colour-magnitude diagrams for cluster 0016+1609

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Because of the existence of a strong galaxies evolution effect in the range $0.4 < z < 0.6$ the extension of the studies of BO effect in clusters with $0.5 < z < 1$ represents a powerful test in the separation between cluster evolution and galaxies evolution. A study of 509 galaxies in 17 rich clusters with $0.2 < z < 0.93$ was done by Rakos 1995 (using the rest frame Stromgren photometry), as a test of color evolution.

In the present paper we determine the colour gradients in 15 clusters of galaxies with $0 < z \leqslant 0.81$ (using photometrical data for 2627 galaxies) and is made the comparison between these gradients and the Butcher-Oemler effect.

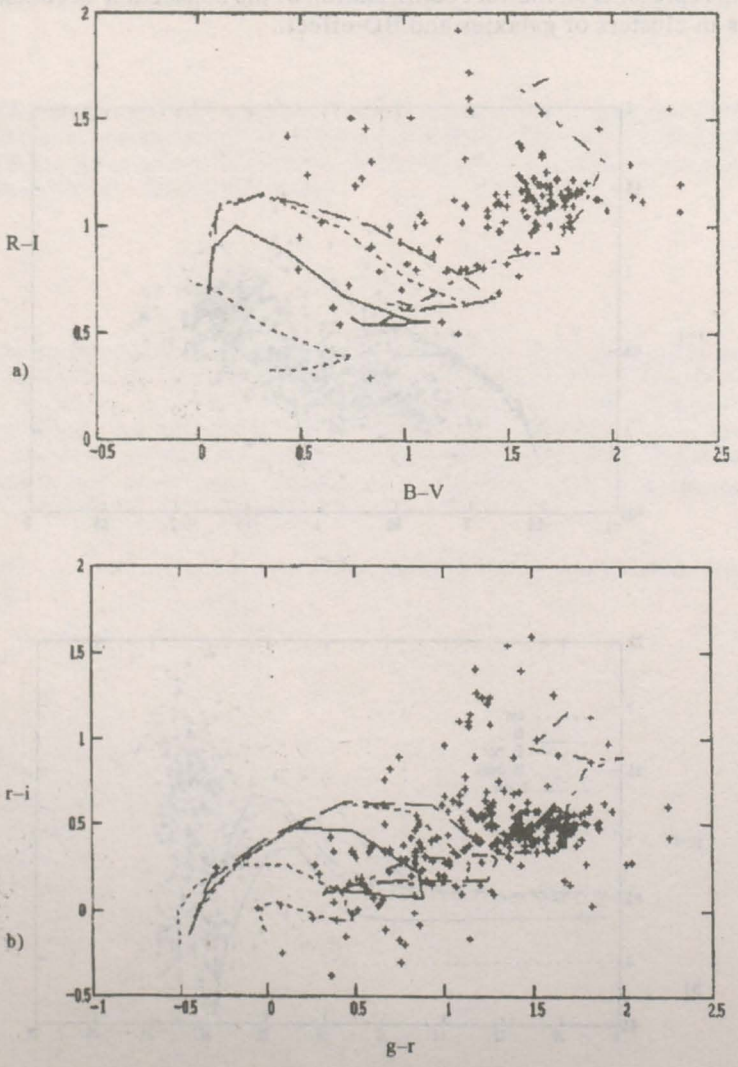



Fig. 2 a, b – The colour-colour diagrams for clusters 1358+6245, 0024+1654

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## 2. THE RELATION BETWEEN COLOUR GRADIENTS AND BUTCHER-OEMLER EFFECT

In the sequel we treat the following galaxy evolution problems :

a) colour gradients in clusters of galaxies;

b) colour evolution (i.e. the mean color <B–V> of galaxies in clusters as function of redshift);

c) Butcher-Oemler effect (i.e. the dependence of the fraction of blue galaxies on redshift);

d) the relation between colour gradients and Butcher-Oemler effect.

a) C–R diagram. We determine the colour gradients in:

i) nearby clusters ($z \cong 0$)

ii) distant clusters ($0.1 < z < 1$).

The colour gradients represent the coefficients $a = dC/dR$ in the $C$–$R$ diagrams, between the colour index ($C$) and the distance of the galaxies to the cluster center ($R$).

We consider the colour index $C = (B–V)$.

The colour gradients can depend on :

– morphological type (T) of the galaxies;

– cluster population (P);

– observational aperture (A);

– redshift (z);

– compactness and richness of the cluster (Bautz-Morgan classes).

*Table 1*

| $z$ | $a_{B-V}$ | $f_B$ | $R_{31}$ | <B–V> | <B–V>$_E$ | <B–V>$_S$ |
|---|---|---|---|---|---|---|
| 0.0235 | – 0.037 | 0.011 | 0.513 | 0.974 | 1.039 | 0.874 |
| 0.0748 | – 0.101 | 0.077 | 0.559 | 1.042 | 1.104 | 0.901 |
| 0.198 | – 0.417 | 0.238 | 0.293 | 1.443 | 1.607 | 0.87 |
| 0.224 | – 0.397 | 0.142 | 0.269 | 1.178 | 1.40 | 1.001 |
| 0.259 | – 0.409 | 0.00 | 0.13 | 1.695 | 1.695 | 0.00 |
| 0.32 | – 0.49 | 0.152 | 0.302 | 1.469 | 1.709 | 1.159 |
| 0.3909 | – 0.581 | 0.147 | 0.384 | 1.855 | 2.07 | 1.314 |
| 0.4069 | – 0.424 | 0,313 | 0.38 | 1.775 | 2.029 | 1.258 |
| 0.4181 | – 0.478 | 0.273 | 0.552 | 1.701 | 2.002 | 1.084 |
| 0.46 | – 0.524 | 0.238 | 0.377 | 1.676 | 2.043 | 1.161 |
| 0.539 | – 0.572 | 0.34 | 0.556 | 1.525 | 1.997 | 0.905 |
| 0.544 | – 0.585 | 0.142 | 0.567 | 1.817 | 2.11 | 1.09 |
| 0.62 | – 0.794 | 0.714 | 0.12 | 1.512 | 1.593 | 1.216 |
| 0.80 | – 0.709 | 0.333 | 0.405 | 1.937 | 2.037 | 1.537 |
| 0.81 | – 0.765 | 0.476 | 0.246 | 1.754 | 1.834 | 1.465 |

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In this paper will be discussed only the dependence of colour gradients on redshift.

In our calculations we had two major difficulties related to:

– the determination of the real cluster population (the separation between field and cluster galaxies );

– the segregation of the galactic populations (E and S morphological type).

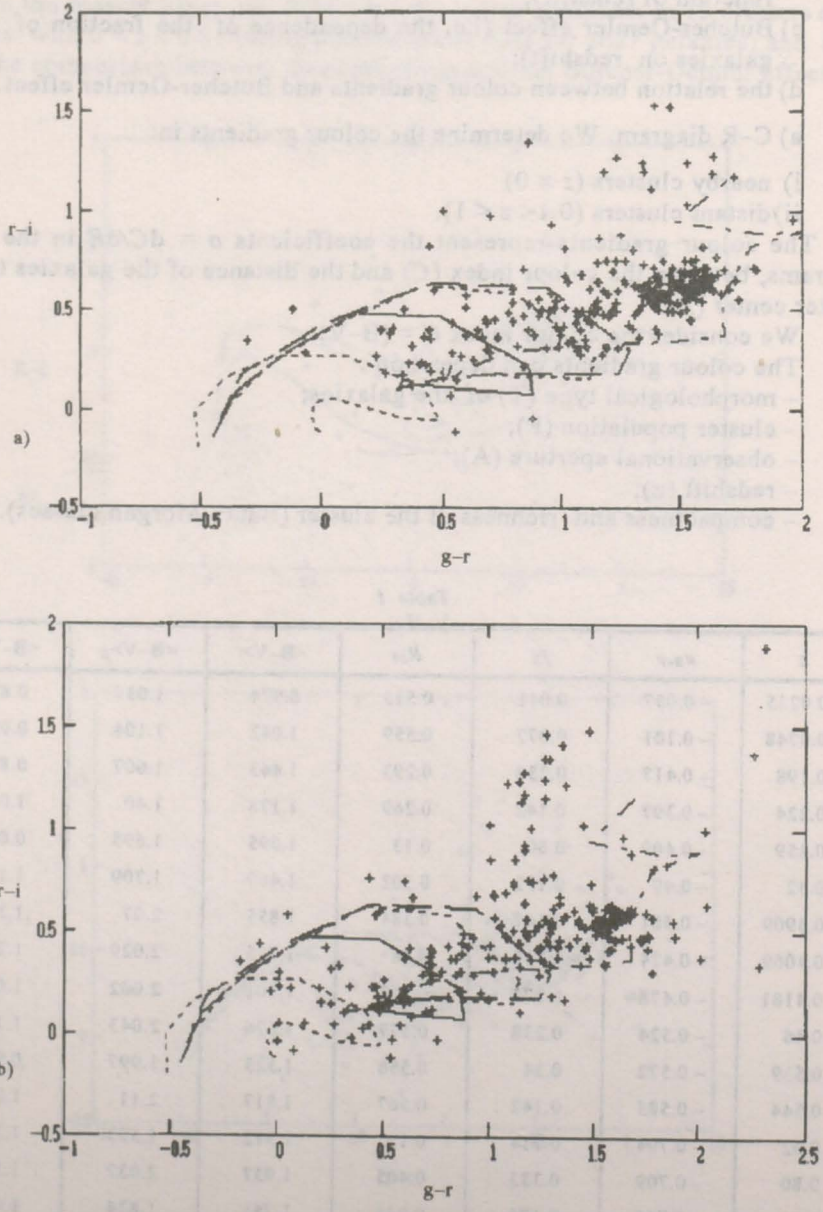



Fig. 3 a, b – The colour-colour diagrams for clusters 0939+4713, 0303+1706

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In order to solve these problems we plotted the observational data of 15 clusters of galaxies on the colour-colour and colour-magnitude diagrams. Also, in these diagrams we plotted the theoretical izo-T lines for redshifts in the range $0 \leqslant z \leqslant 1.4$ (Rocca-Volmerange, Guiderdoni's spectrophotometrical

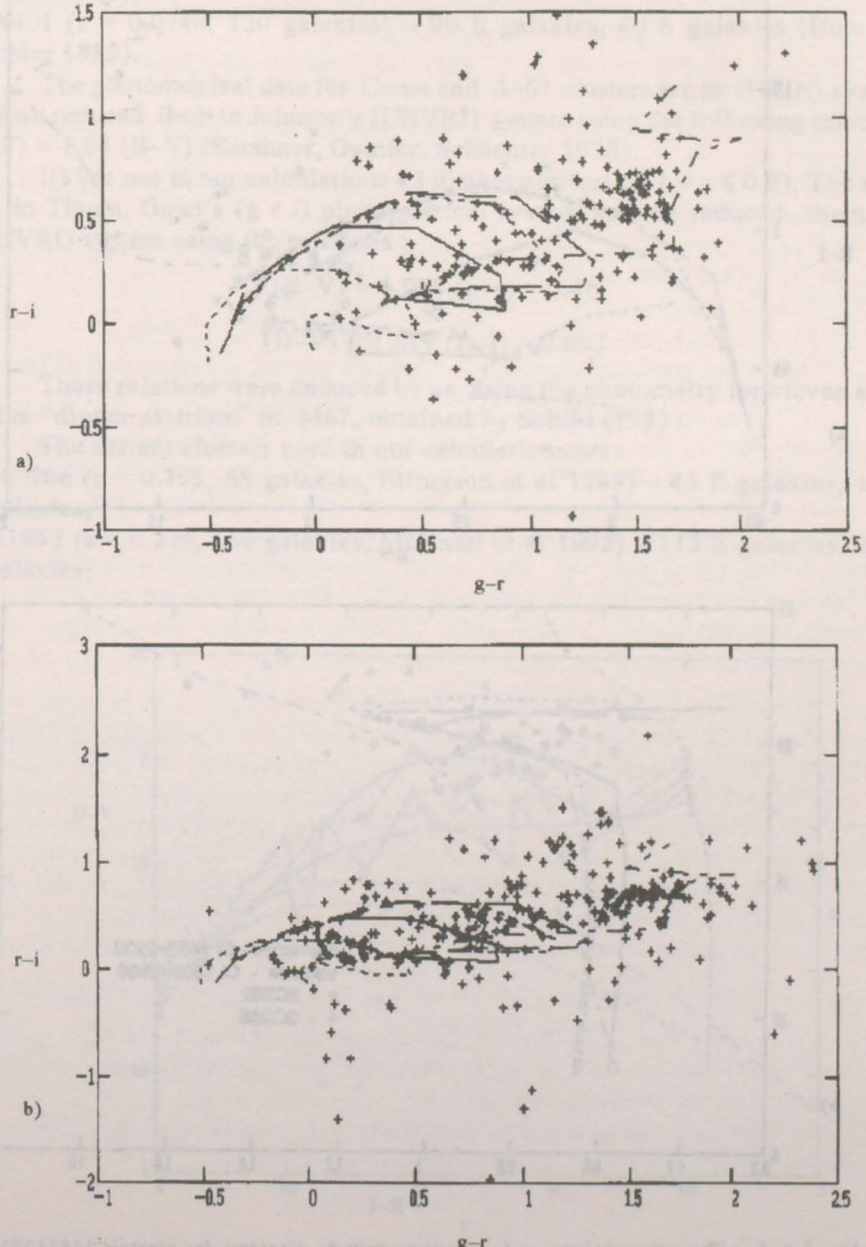



Fig. 4 a, b – The colour-colour diagrams for clusters 3C295, 1601+4253

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model with evolution 1988). In Figs 1–5 are presented the diagrams for some clusters used in our calculations (+ represents the galaxies in clusters). The izo-T

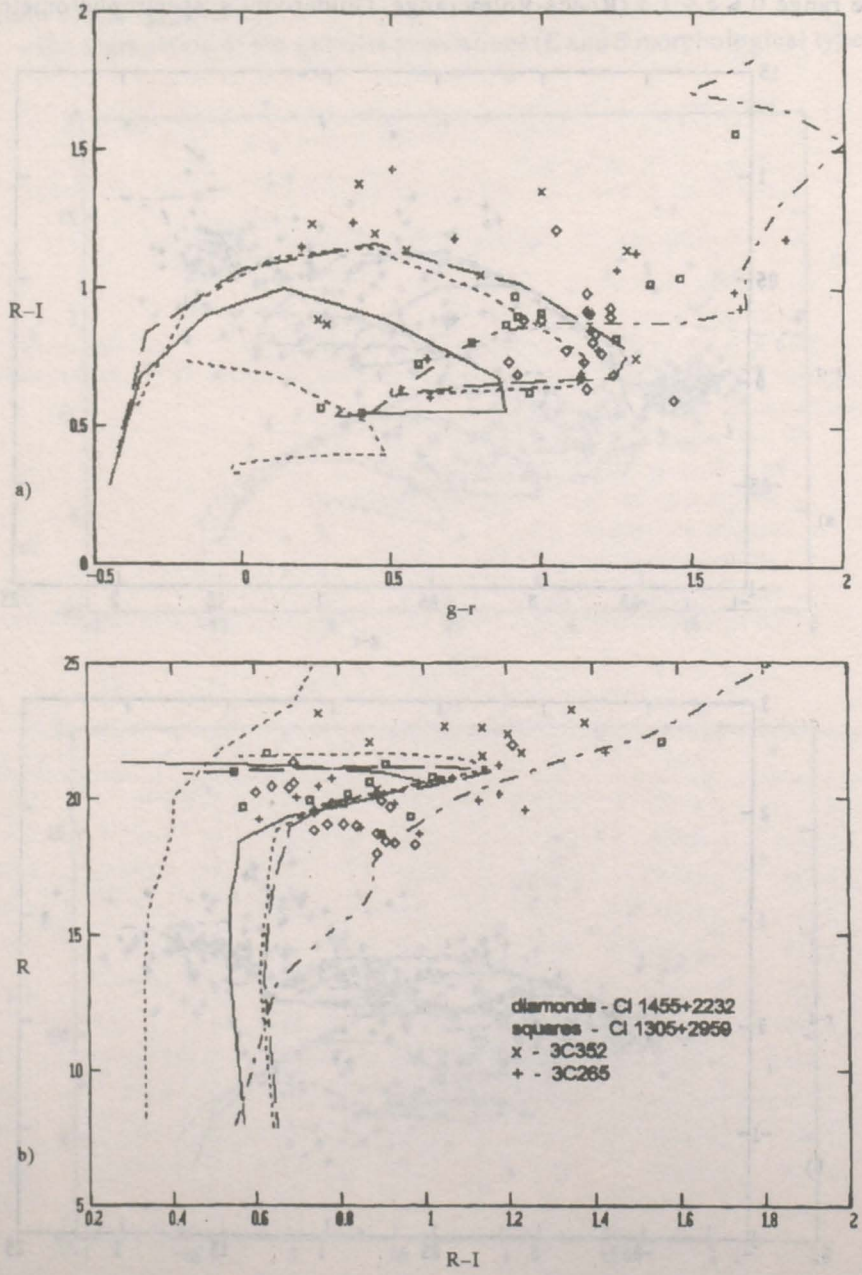



Fig. 5 a, b – The colour-colour and colour-magnitude diagrams for clusters 1455+2232, 1305+2959, 3C352, 3C265

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lines are designed as follows: dot line-Im type, solid line-S type, dot line-E hot type, dash line-E cold type, dadot line-burst (from left to right, in colour-colour diagrams).

i) We have photometrical data for the following nearby clusters :
– Coma ($z$ = 0.0235 , 306 galaxies) – 183 E galaxies, 123 S galaxies (Butcher, Oemler 1985);
– A401 ($z$ = 0.0748, 130 galaxies) – 90 E galaxies, 40 S galaxies (Butcher, Oemler 1985).

The photometrical data for Coma and A401 clusters are in (UJFN) system and we reduced them to Johnson's (UBVRI) system using the following relation: (J–F) = 1.04 (B–V) (Kirshner, Oemler, Schechter 1978).

ii) We use in our calculations 13 distant clusters ($0.1 < z \leqslant 0.8$). The data are in Thuan, Gunn's (g r i) photometrical system and we reduced them to (UBVRI) system using the relations :

$$(B–V) = 1.094\ (g–r) + 0.46$$

$$(B–V) = 1.817\ (R–I) - 0.062$$

These relations were deduced by us using the photometry for eleven stars in the "dipper asterism" in M67, obtained by Schild (1983).

The distant clusters used in our calculations are:
– 3C 206 ($z$ = 0.198, 65 galaxies, Ellingson et al 1989) – 45 E galaxies, 11 S galaxies;
– A1942 ($z$ = 0.224, 166 galaxies, Molinari et al 1992) – 115 E galaxies, 27 S galaxies;

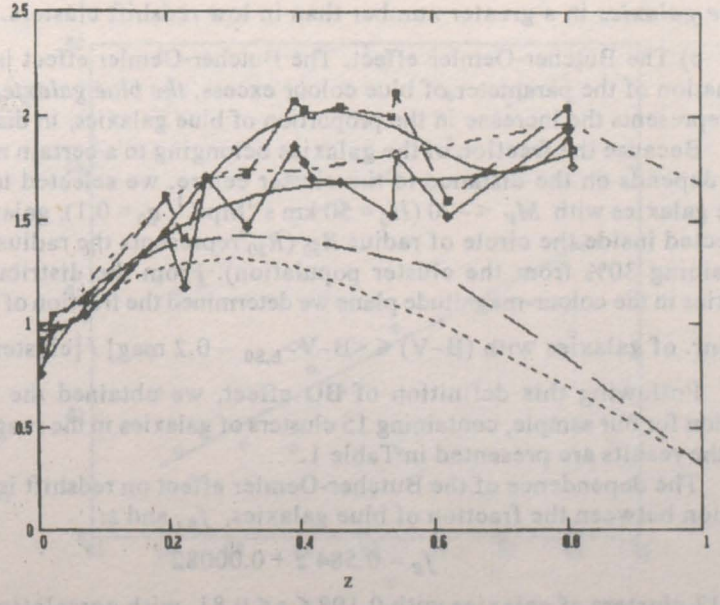



Fig. 6 – The (B–V) – $z$ diagram

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- Cl 1455 + 2232 ($z$ = 0.25, 19 galaxies, Schild 1984 ) – 19 E galaxies;
- Cl 1358 + 6245 ($z$ = 0.32 , 128 galaxies, Luppino et al 1991) – 62 E galaxies, 48 S galaxies;
- Cl 0024 + 1654 ($z$ = 0.39, 274 galaxies, Dressler and Gunn 1992) – 164 E galaxies, 65 S galaxies;
- Cl 0939 + 4713 ($z$ = 0.4069, 274 galaxies, Dressler and Gunn 1992, Dressler et al 1994) – 160 E galaxies, 65 S galaxies;
- Cl 0303 + 1706 ($z$ = 0.4181, 252 galaxies, Dressler and Gunn 1992) – 148 E galaxies, 72 S galaxies;
- 3C 295 ($z$ = 0.46, 200 galaxies, Dressler and Gunn 1992) – 98 E galaxies, 70 S galaxies;
- Cl 1601 + 4253 ($z$ = 0.5391, 307 galaxies, Dressler and Gunn 1992) – 139 E galaxies, 106 S galaxies;
- Cl 0016 + 1609 ($z$ = 0.5455, 467 galaxies, Dressler and Gunn 1992) – 285 E galaxies, 115 S galaxies;
- Cl 1305 + 2959 ($z$ = 0.62, 13 galaxies, Schild 1984);
- 3C 352 + 3C 265 ($z$ = 0.80, $z$ = 0.81, 26 galaxies, Schild 1984).

The colour gradients obtained for these 15 clusters are presented in Table 1.

b) Colour evolution. The column <B–V> in Table 1 represents the ***intrinsic evolution effect*** of the galaxies in clusters. The smoothed curve, designed with the lower continuous track in Fig.6, is obtained with a third degree polynomial. This curve fit is in good agreement with the theoretical burst track of Rocca-Volmerange, Guiderdoni's spectrophotometrical model. Concerning the mean colour index for E galaxies, the curve fit is obtained with a fourth degree polynomial (the upper continuous track in Fig. 6) and in the range $0.4 < z < 0.6$ this one presents a peak. This effect is probably a result of the presence of the *active galaxies* in a greater number than in low redshift clusters.

c) The Butcher-Oemler effect. The Butcher-Oemler effect is based on the estimation of the parameter of blue colour excess, ***the blue galaxies fraction*** ($f_B$), and represents the increase in the proportion of blue galaxies, in distant cluster.

Because the fraction of the galaxies belonging to a certain morphological type depends on the distance to the cluster centre, we selected in each cluster those galaxies with $M_V < -20$ ($H_0 = 50$ km s$^{-1}$Mpc$^{-1}$, $q_0 = 0.1$), galaxies which are projected inside the circle of radius $R_{30}$ ($R_{30}$ represents the radius of the circle containing 30% from the cluster population). From the distribution of these galaxies in the colour-magnitude plane we determined the fraction of blue galaxies:

$f_B$=[nr. of galaxies with (B–V) $\leqslant$ <B–V>$_{E,S0}$ – 0.2 mag] / [cluster population]

Following this definition of BO effect, we obtained the blue galaxies fraction for our sample, containing 15 clusters of galaxies in the range $0 < z \leqslant 0.81$, and the results are presented in Table 1.

The dependence of the Butcher-Oemler effect on redshift is shown by the relation between the fraction of blue galaxies, $f_B$, and $z$:

$$f_B = 0.584\, z + 0.00082$$

(for 13 clusters of galaxies with $0.198 \leqslant z \leqslant 0.81$, with correlation coefficient $r = 0.644$).

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We have to mention that Butcher-Oemler 1984 obtained $df_B/dz = 0.52$ (for 21 clusters with $0.1<z<0.55$), Newberry et al.1988 obtained $df_B/dz = 0.6$ (for 62 galaxies in seven dense BO clusters with redshifts $0.2 \leqslant z \leqslant 0.38$) and Rakos, Schombert 1995 obtained $df_B/dz = 0.653$ (for 17 clusters of galaxies with $0.183 \leqslant z \leqslant 0.927$, considering that the maximum number of blue galaxies is placed in the distance between 0.5 and 1 Mpc from the center, not only in the central region of the cluster).

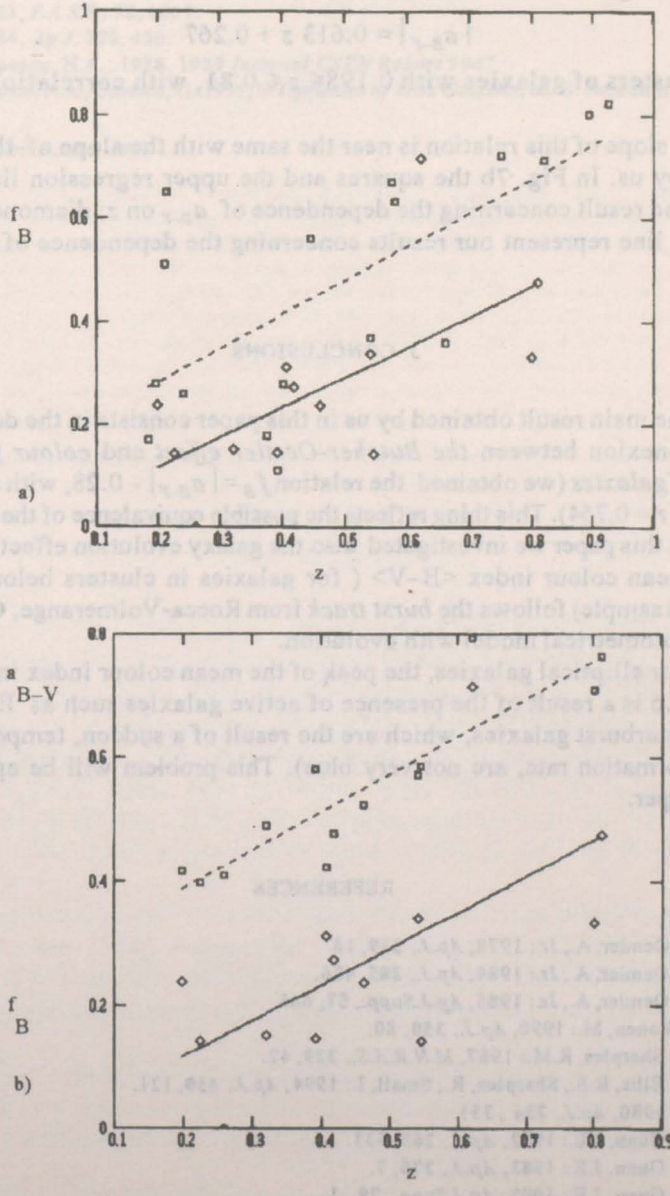



Fig. 7 a, b – The $f_B - z$ and $a_{B-V} - z$ diagrams.

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In Fig. 7a, diamonds and the lower regression line represent our data and results concerning the dependence of $f_b$ on $z$; squares and the upper regression line represent Rakos's data and results concerning the same problem.

d) Colour gradients related with the Butcher-Oemler effect. We obtained the following relation concerning the dependence on $z$ of the colour gradients in clusters of galaxies:

$$|a_{B-V}| = 0.613\ z + 0.267$$

(for 13 clusters of galaxies with $0.198 \leqslant z \leqslant 0.81$, with correlation coefficient $r = 0.896$).

The slope of this relation is near the same with the slope of the BO effect obtained by us. In Fig. 7b the squares and the upper regression line represent our data and result concerning the dependence of $a_{B-V}$ on $z$; diamonds and lower regression line represent our results concerning the dependence of $f_B$ on $z$.

## 3. CONCLUSIONS

– The main result obtained by us in this paper consists in the determination of the connexion between *the Butcher-Oemler effect* and *colour gradients in clusters of galaxies* (we obtained the relation $f_B = |a_{B-V}| - 0.28$, with a correlation coefficient $r = 0.754$). This thing reflects the possible equivalence of the phenomena.

– In this paper we investigated also the galaxy evolution effect. We showed that the mean colour index <B–V> ( for galaxies in clusters belonging to the considered sample) follows the *burst track* from Rocca-Volmerange, Guiderdoni's spectrophotometrical model with evolution.

– For elliptical galaxies, the peak of the mean colour index in the interval $0.4 < z < 0.6$ is a result of the presence of active galaxies such as E+A galaxies (the post-starburst galaxies, which are the result of a sudden, temporary rise in the star formation rate, are not very blue). This problem will be approached in another paper.

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# STUDY OF THE ECLIPSING BINARY GO CYGNI BASED ON OBSERVATIONS OF BUCHAREST AND ATHENS OBSERVATORIES FROM 1985 TO 1994

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*Abstract.* The principal aim of this work is to study the eclipsing binary system GO Cygni for which a series of photoelectric observations have been carried out for ten years, 1985–1994, either at the Athens or at the Bucharest Observatories. The unpublished data obtained from 1989 to 1992 are given and the corresponding light curves of the system are presented while a general discussion concerning some of their features is made. A shift of the order of 0.082 found in the V light curves of 1989–90 & 92 and those of 1993–94 can be due to the variability of the comparison star BD + 35° 4197, which has to be checked; while some peculiar features of its light curves may be due to the activity of the system. Moreover, the observed variations in the period of GO Cygni are discussed.

*Key words:* stellar photometry, close binary systems.

## 1. INTRODUCTION

The variability of GO Cygni (BD + 34° 4095) was discovered photographically by Schneller (1928) and the system was classified as a short-period eclipsing binary by Kukarkin (1929). Szczyrbak (1931) gave its first elements, considering the system to be of Algol type; but, Kukarkin's visual observations made between 1929–31 (Kukarkin, 1932) and which were subsequently analyzed by Kopal (1932) showed a β Lyrae type variation. From its early photographic (Liau, 1935) and visual (Pierce, 1939) observations contradictory results were found for both its light curve shape and the eccentricity of its orbit (for details see Ovenden, 1954).

Photoelectric observations and light curves analyses of GO Cyg have been made by Ovenden (1954), Popper (1957), Mannino (1963), Rovithis et al. (1990), Sezer et al. (1992) and Rovithis-Livaniou et al. (1996). Details concerning these observations and the corresponding light curves analyses are given in Tables 1, 2 and 3.

The first spectroscopic elements of the system were published by Pearce (1933) and a new spectroscopic orbit was obtained by Holmgren using the cross-correlation techniques (see e. g. Sezer et al., 1992).



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*Table 1*

A list of the photometric observations of GO Cygni

| Year of obs. | Comp. & Checking Star | Reference |
|---|---|---|
| 1950–1951 | $+35^0$ 4197 , $+34^0$ 4098 | Ovenden, 1954 |
| 1959–1962 | $+35^0$ 4197 , $+34^0$ 4098 | Mannino, 1963 |
| 1984–1985 | $+35^0$ 4197 , $+34^0$ 4098 | Sezes et al., 1992 |
| 1985 | HD 197292 , HD 197346 | Rovithis et al., 1990 |
| 1993–1994 | $+35^0$ 4197 , $+34^0$ 4098 | Rovithis-Livaniou et al., 1996 |
| 1989–1990, 1992 | $+35^0$ 4197 , $+34^0$ 4098 | Present study |

*Table 2*

Results from the light curves analysis of GO Cygni

| Method | Oveden (R–M) | | Mannino (R–M) | Rovithis (Kopal) | | Sezer (W–D) | |
|---|---|---|---|---|---|---|---|
| | V | B | | V | B | V | B |
| $r_1$ | 0.513 | – | | 0.460 | 0.458 | | |
| $r_2$ | 0.333 | – | | 0.255 | 0.257 | | |
| k | 0.67 | – | | 0.55 | 0.56 | | |
| i | 73.0 | – | 63.00 | 74.90 | 74.20 | 79.616 | 79.203 |
| $u_1$ | 1.0 | 0.80 | | 0.41 | 0.49 | 0.45 | 0.55 |
| $u_2$ | 1.0 | 0.80 | | 0.61 | 0.73 | 0.62 | 0.79 |
| $L_1$ | 0.93 | – | | 0.970 | 0.973 | | |
| $L_2$ | 0.70 | – | | 0.030 | 0.027 | | |
| $\iota_1$ | | – | | 1.20* | 1.10* | | |
| $\iota_2$ | | | | 0.40* | 0.41* | | |
| q | 0.53 | | | 0.53 | | 0.428 | |
| $a_1$ | | | 0.454 | | | | |
| $a_2$ | | | 0.408 | | | | |
| $b_1$ | | | 0.424 | | | | |
| $b_2$ | | | 0.382 | | | | |
| e | 0 | | 0 | 0 | | 0 | |
| $T_1$ | | | | 10350 | | 10350 | |
| $T_2$ | | | | 6114 | | 5937 | |
| $\Omega_1$ | | | | 2.735 | | 2.729* | |
| $\Omega_2$ | | | | 2.944 | | 2.908 | |
| $r_1$, pole | | | | 0.427 | | 0.428 | |
| $r_1$, side | | | | 0.455 | | 0.456 | |
| $r_1$, back | | | | 0.482 | | 0.483 | |
| $r_2$, pole | | | | 0.254 | | 0.257 | |
| $r_2$, side | | | | 0.261 | | 0.265 | |
| $r_2$, back | | | | 0.278 | | 0.283 | |

* : assumed and modified gravity darkening coefficients

Indices 1&2 are referred to primary and secondary components, respectively.

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*Table 3.A.*

GO Cygni elements obtained for Athens and Bucharest mean light curves using WOOD model ($T_h$=10350K)

Variants for all data:
(1) improve of ( i, k = $r_h/r_c$ , $r_h$);
(2) improve of ( i, k, $r_h$ ,$T_c$ );
(3) using step (2) improve of (i, k,$r_h$ ,$T_h$ ,$T_c$).

*3.A.a.* GO Cyg results, Athens

| Parameter | V1 | V2 | V3 | B1 | B2 | B3 |
|---|---|---|---|---|---|---|
| i | 77.42 | 77.40 | 77.40 | 77.26 | 77.21 | 78.96 |
| | ± 0.003 | ± 0.004 | ± 0.002 | ± 0.003 | ± 0.003 | ± 0.003 |
| $u_h$ | 0.41 | 0.41 | 0.41 | 0.41 | 0.41 | 0.47 |
| $u_c$ | 0.61 | 0.61 | 0.61 | 0.61 | 0.61 | 0.59 |
| k | 0.569 | 0.569 | 0.60 | 0.584 | 0.584 | 0.547 |
| | ± 0.004 | ± 0.004 | ± 0.006 | ± 0.004 | ± 0.004 | ± 0.007 |
| $r_h$ | 0.490 | 0.490 | 0.491 | 0.472 | 0.473 | 0.475 |
| | ± 0.009 | ± 0.008 | ± 0.010 | ± 0.008 | ± 0.009 | ± 0.009 |
| $a_h$ | 0.540 | 0.540 | 0.547 | 0.515 | 0.516 | 0.520 |
| $b_h$ | 0.493 | 0.493 | 0.498 | 0.475 | 0.475 | 0.478 |
| $c_h$ | 0.447 | 0.448 | 0.456 | 0.436 | 0.436 | 0.438 |
| $r_c$ | 0.27 | 0.27 | 0.31 | 0.27 | 0.27 | 0.26 |
| $a_c$ | 0.294 | 0.294 | 0.332 | 0.291 | 0.291 | 0.273 |
| $b_c$ | 0.277 | 0.277 | 0.306 | 0.274 | 0.274 | 0.259 |
| $c_c$ | 0.268 | 0.268 | 0.293 | 0.266 | 0.266 | 0.252 |
| $\beta_h$ | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 |
| $\beta_c$ | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 |
| $T_h$ | 10350 | 10350 | 13813 | 10350 | 10350 | 11547 |
| | | | ±890 | | | ±942 |
| $T_c$ | 5348 | 5348 | 6150 | 5988 | 5988 | 6150 |
| | ±335 | ±335 | | ±293 | ±293 | |
| q | 0.53 | 0.53 | 0.53 | 0.53 | 0.53 | 0.53 |
| $L_h$ | 0.97 | 0.97 | 0.95 | 0.96 | 0.96 | 0.97 |
| $L_c$ | 0.026 | 0.026 | 0.045 | 0.030 | 0.030 | 0.024 |
| $(O-C)^2$ | 0.0186 | 0.0186 | 0.0300 | 0.0180 | 0.0181 | 0.0195 |

Many minima times for GO Cygni can be found in the literature, while studies of its period have been made by Purgathofer & Prochazka (1967), Cester et al. (1979), Hall & Louth (1990), Jones et al. (1994) and Rovithis-Livaniou et al. (1996).

Here, we shall present all data of GO Cygni obtained at Bucharest and Athens Observatories and discuss all available data concerning the system and made so far.

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*Table 3.A.b.*

GO Cyg results, Bucharest 1993–1994

| Parameter | V1 | V2 | V3 | B1 | B2 | B3 |
|---|---|---|---|---|---|---|
| i | 77.92 | 77.6 | 75.0 | 78.30 | 78.15 | 79.67 |
| | ±0.004 | ±0.003 | ±0.002 | ±0.004 | ±0.003 | ±0.004 |
| $u_h$ | 0.41 | 0.41 | 0.41 | 0.41 | 0.41 | 0.47 |
| $u_c$ | 0.61 | 0.61 | 0.61 | 0.61 | 0.61 | 0.59 |
| k | 0.536 | 0.575 | 0.648 | 0.577 | 0.593 | 0.563 |
| | ±0.005 | ±0.004 | ±0.006 | ±0.005 | ±0.005 | ±0.007 |
| $r_h$ | 0.515 | 0.478 | 0.464 | 0.487 | 0.470 | 0.468 |
| | ±0.010 | ±0.009 | ±0.010 | ±0.010 | ±0.011 | ±0.010 |
| $a_h$ | 0.574 | 0.522 | 0.507 | 0.536 | 0.513 | 0.510 |
| $b_h$ | 0.517 | 0.480 | 0.469 | 0.490 | 0.473 | 0.471 |
| $c_h$ | 0.462 | 0.439 | 0.432 | 0.446 | 0.434 | 0.433 |
| $r_c$ | 0.27 | 0.27 | 0.30 | 0.28 | 0.28 | 0.26 |
| $a_c$ | 0.291 | 0.289 | 0.325 | 0.297 | 0.296 | 0.276 |
| $b_c$ | 0.274 | 0.273 | 0.300 | 0.279 | 0.278 | 0.262 |
| $c_c$ | 0.266 | 0.264 | 0.288 | 0.270 | 0.269 | 0.255 |
| $\beta_h$ | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 |
| $\beta_c$ | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 |
| $T_h$ | 10350 | 10350 | 8749 | 10350 | 10350 | 10367 |
| | | | ±231 | | | 673 |
| $T_c$ | 6301 | 6747 | 6150 | 6012 | 6289 | 6150 |
| | ±273 | ±166 | | ±355 | ±2281 | |
| q | 0.53 | 0.53 | 0.53 | 0.53 | 0.53 | 0.53 |
| $L_h$ | 0.95 | 0.92 | 0.89 | 0.96 | 0.95 | 0.96 |
| $L_c$ | 0.049 | 0.070 | 0.100 | 0.031 | 0.041 | 0.035 |
| $(O-C)^2$ | 0.0291 | 0.0253 | 0.0250 | 0.0308 | 0.0310 | 0.0333 |

## 2. OBSERVATIONS OF GO CYGNI AND ITS PHYSICAL PARAMETERS

Our first photoelectric observations of GO Cygni were made during 1985 at the Kryonerion Astronomical Station of the National Observatory of Athens, Greece; the system was re-observed at the Bucharest Observatory during 1989, 1990, 1992 and 1993–94. The corresponding mean light curves of our "first" (1985) and "last" (1993–94) observations are presented in Figures 1 & 2, for comparison.

The foregoing mentioned observations of GO Cyg have been analyzed (Rovithis-Livaniou et al., 1996) using Wood's model, with input values those found by Rovithis et al. (1990). From this work, and in order to be able to compare all elements of GO Cyg, we recall the results obtained for mass-ratio $q = 0.426$ only. These are presented in Table 3, where the gravity darkening coefficients were fixed to 0.25, considering convective envelopes for both components.

Meanwhile, as it has already been mentioned, the system was also observed during 1989, 1990 & 1992. These unpublished observations of GO Cygni are

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*Table 3.B.*

GO Cyg elements obtained for Athens and Bucharest mean light curves using WOOD model ($T_h$ = 11200 K)

. Variants for all data:
(1) improve of (i, k = $r_h/r_c$, $r_h$);
(2) improve of (i, k, $r_h$, $T_c$);
(3) using step (2) improve of (i, k, $r_h$, $T_h$, $T_c$).

*3.B.a.* GO Cyg results, Athens.

| Parameter | V1 | V2 | V3 | B1 | B2 | B3 |
|---|---|---|---|---|---|---|
| i | 77.4 | 77.4 | 77.4 | 77.3 | 77.3 | 79.1 |
| | ±0.003 | ±0.004 | ±0.002 | ±0.003 | ±0.003 | ±0.003 |
| $u_h$ | 0.41 | 0.41 | 0.41 | 0.41 | 0.41 | 0.47 |
| $u_c$ | 0.61 | 0.61 | 0.61 | 0.61 | 0.61 | 0.59 |
| k | 0.54 | 0.54 | 0.60 | 0.56 | 0.56 | 0.54 |
| | ±0.004 | ±0.004 | ±0.006 | ±0.004 | ±0.004 | ±0.007 |
| $r_h$ | 0.49 | 0.49 | 0.49 | 0.47 | 0.47 | 0.47 |
| | ±0.009 | ±0.008 | ±0.010 | ±0.008 | ±0.009 | ±0.009 |
| $a_h$ | 0.540 | 0.540 | 0.547 | 0.515 | 0.516 | 0.520 |
| $b_h$ | 0.493 | 0.493 | 0.498 | 0.475 | 0.475 | 0.478 |
| $c_h$ | 0.447 | 0.448 | 0.456 | 0.436 | 0.436 | 0.438 |
| $r_c$ | 0.27 | 0.27 | 0.31 | 0.27 | 0.27 | 0.26 |
| $a_c$ | 0.294 | 0.294 | 0.332 | 0.291 | 0.291 | 0.273 |
| $b_c$ | 0.277 | 0.277 | 0.306 | 0.274 | 0.274 | 0.259 |
| $c_c$ | 0.268 | 0.268 | 0.293 | 0.266 | 0.266 | 0.252 |
| $\beta_h$ | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 |
| $\beta_c$ | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 |
| $T_h$ | 11200 | 11200 | 14506 | 11200 | 11200 | 12836 |
| | | | ±991 | | | ±992 |
| $T_c$ | 5661 | 5661 | 7176 | 6355 | 6355 | 6693 |
| | ±343 | ±346 | | ±303 | ±303 | |
| q | 0.53 | 0.53 | 0.53 | 0.53 | 0.53 | 0.53 |
| $L_h$ | 0.97 | 0.97 | 0.95 | 0.96 | 0.96 | 0.97 |
| $L_c$ | 0.026 | 0.026 | 0.045 | 0.030 | 0.030 | 0.024 |
| $(O-C)^2$ | 0.0182 | 0.0182 | 0.0148 | 0.0175 | 0.0175 | 0.0191 |

given in Table 4 (for the V filter) with the corresponding light curves presented in Figures 3, 4 & 5 for 1989, 1990 & 1992, respectively; while in Figure 6 they are presented all together. The phases in these figures have been computed using S. A. C. 1989 ephemeris formula:

$$\text{Min I} = 2446351.328 + 0^d\,71777632\ E$$

and magnitude differences are *comp-var.*

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*Table3.B.b.*

GO Cyg results, Bucharest 1993–1994.

| Parameter | V1 | V2 | V3 | B1 | B2 | B3 |
|---|---|---|---|---|---|---|
| i | 78.0 | 77.6 | 74.7 | 78.3 | 78.1 | 79.9 |
| | ±0.004 | ±0.003 | ±0.002 | ±0.004 | ±0.003 | ±0.004 |
| $u_h$ | 0.41 | 0.41 | 0.41 | 0.41 | 0.41 | 0.47 |
| $u_c$ | 0.61 | 0.61 | 0.61 | 0.61 | 0.61 | 0.59 |
| k | 0.50 | 0.50 | 0.61 | 0.55 | 0.57 | 0.54 |
| | ±0.005 | ±0.004 | ±0.006 | ±0.005 | ±0.005 | ±0.007 |
| $r_h$ | 0.51 | 0.48 | 0.47 | 0.49 | 0.47 | 0.47 |
| | ±0.010 | ±0.009 | ±0.010 | ±0.010 | ±0.011 | ±0.010 |
| $a_h$ | 0.574 | 0.522 | 0.507 | 0.536 | 0.513 | 0.510 |
| $b_h$ | 0.517 | 0.480 | 0.469 | 0.490 | 0.473 | 0.471 |
| $c_h$ | 0.462 | 0.439 | 0.432 | 0.446 | 0.434 | 0.433 |
| $r_c$ | 0.27 | 0.27 | 0.30 | 0.28 | 0.28 | 0.26 |
| $a_c$ | 0.291 | 0.289 | 0.325 | 0.297 | 0.296 | 0.276 |
| $b_c$ | 0.274 | 0.273 | 0.300 | 0.279 | 0.278 | 0.262 |
| $c_c$ | 0.266 | 0.264 | 0.288 | 0.270 | 0.269 | 0.255 |
| $\beta_h$ | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 |
| $\beta_c$ | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 |
| $T_h$ | 11200 | 11200 | 10579 | 11200 | 11200 | 11448 |
| | | | ±289 | | | ±753 |
| $T_c$ | 6718 | 7176 | 7176 | 6408 | 6693 | 6693 |
| | ±294 | ±181 | | ±369 | ±289 | |
| q | 0.53 | 0.53 | 0.53 | 0.53 | 0.53 | 0.53 |
| $L_h$ | 0.95 | 0.92 | 0.89 | 0.96 | 0.95 | 0.96 |
| $L_c$ | 0.049 | 0.070 | 0.100 | 0.031 | 0.041 | 0.035 |
| $(O-C)^2$ | 0.0294 | 0.0251 | 0.0252 | 0.0308 | 0.0310 | 0.0330 |

## 3. ORBITAL PERIOD STUDY OF GO CYGNI SYSTEM

The first light elements of GO Cyg were given by Szczyrbak (1932):

$$\text{Min I} = \text{Hel JD } 2426509.467 + 0^d.717767\ E$$

Later, Purgathofer & Prochazka (1967) noticed some variation in its orbital period and gave the following linear and quadratic light elements:

$$\text{Min I} = \text{Hel JD } 2433930.40561 + 0^d.71776382\ E$$

and

$$\text{Min I} = \text{Hel JD } 2433930.40614 + 0^d.71776314\ E + 0^d.108\times10^{-9}\ E^2$$

respectively.

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*Table 3.C.*

GO Cyg elements obtained for Athens and Bucharest mean light curves using Wilson-Devinney method (ROCHE Model) ($T_1$ = 10350 K) .The results are obtained using our program (full reflection method) using differential correction (the same method as in WOOD). The parameters to be improved were ($i, C_1, C_2, T_2$) where $C_1, C_2$-Kopal, Shapley potentials ($C_i = 2 * \Omega_i / (1+q)$)

| Parameter | B(Ath) | V(Ath) | B(Buc) | V(Buc) |
|---|---|---|---|---|
| i | 77.00 | 78.00 | 78.00 | 76.00 |
| | ±0.011 | ±0.008 | ±0.007 | ±0.007(rad)3.920. |
| $C_1$ | 3.934 | 3.942 | 3.943 | 00 |
| | ±0.008 | ±0.005 | ±0.036 | ±0.034 |
| $C_2$ | 4.129 | 4.129 | 4.127 | 4.090 |
| | ±0.008 | ±0.003 | ±0.011 | ±0.010 |
| $T_1$ | 10350.00 | 10350.00 | 10350.00 | 10350.00 |
| $T_2$ | 5937.00 | 6089.00 | 6248.00 | 6107.00 |
| | ±118.00 | ±282.00 | ±209.00 | ±259.00 |
| $u_1$ | 0.55 | 0.45 | 0.55 | 0.45 |
| $u_2$ | 0.79 | 0.62 | 0.79 | 0.62 |
| $\beta_1$ | 0.25 | 0.25 | 0.25 | 0.25 |
| $\beta_2$ | 0.25 | 0.25 | 0.25 | 0.25 |
| $w_1$ (albedo) | 0.5 | 0.5 | 0.5 | 0.5 |
| $w_2$ (albedo) | 0.5 | 0.5 | 0.5 | 0.5 |
| $L_1$ | 0.941 | 0.936 | 0.953 | 0.934 |
| $L_2$ | 0.059 | 0.064 | 0.047 | 0.066 |
| $r_1$, pole | 0.425 | 0.425 | 0.425 | 0.428 |
| $r_1$, back | 0.475 | 0.475 | 0.475 | 0.479 |
| $r_1$, side | 0.451 | 0.448 | 0.448 | 0.453 |
| $r_2$, pole | 0.263 | 0.263 | 0.253 | 0.266 |
| $r_2$, back | 0.302 | 0.302 | 0.278 | 0.308 |
| $r_2$, side | 0.270 | 0.270 | 0.260 | 0.276 |
| $(O-C)^2$ | 0.0106 | 0.0128 | 0.0277 | 0.0265 |

Moreover, Cester et al. (1979) found:

$$\text{Min I} = \text{Hel JD } 2433930.4055 + 0.71776434\ E$$

and

$$\text{Min I} = \text{Hel JD } 2433930.4061 + 0.7177636\ E + 7\times10^{-11}\ E^2$$

while Sezer et al. (1985), improving the quadratic ephemeris to:

$$\text{Min I} = \text{Hel JD } 2433930.4060 + 0.71776331\ E + 0.113\times10^{-9}\ E^2$$

found an increase in the period of the system about 0.99±0.11 sec per century, and calculated and proposed the new linear ephemeris:

$$\text{Min I} = \text{Hel JD } 2445865.4056 + 0.71776707\ E$$

Finally, Jones et al. (1994) improved the quadratic ephemeris of Purgathofer & Prochazka to:

$$\text{Min I} = 2433930.40535 + 0.71776285\ E + 1.531\times10^{-10}\ E^2$$

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and found a periodicity of 37.8 years; but, they have not taken into account all minima times considering many of them as discordant.

From all available in the literature minima times of GO Cyg the up today (O–C) diagram of the system was constructed and is presented in Figure 7. It consists of 6 photographic, 31 visual and 84 photoelectric minima times, and it is described by a quadratic (dashed line) and a third order (continuous line) polynomial. The third order was found to give a much better approximation than one can notice from Table 5, where the coefficients of these polynomials and their rms errors are given; moreover, a scale factor equal to 37 000 has been

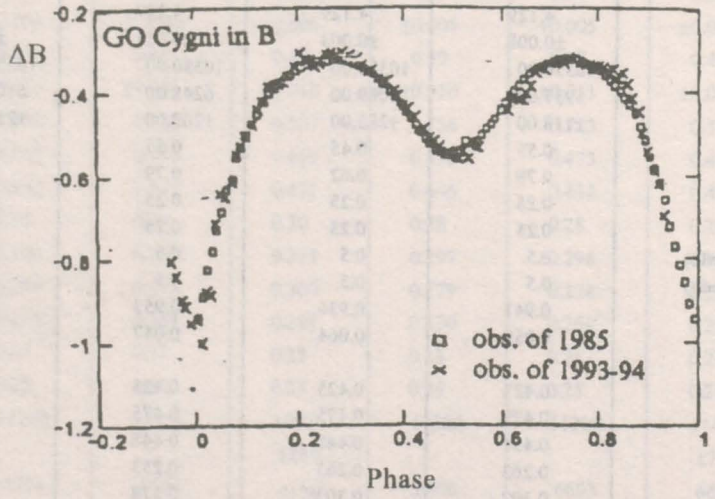



Fig. 1 – Mean light curves of GO Cygni in B colour, based on our photoelectric observations carried out at Athens & Bucharest Observatories in 1985 & 1993–94, respectively.

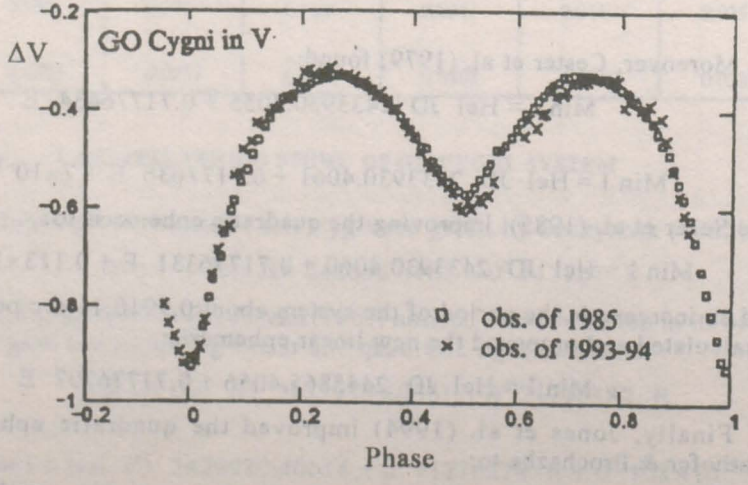



Fig. 2 – Same as Fig. 1 but in V colour.

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used to keep the coefficients values small enough. For their computation different weights have been given to the data: 0.1 for the visual, 0.3 for the photographic and 1 for the photoelectric ones. Then, using the new method proposed and developed by Kalimeris et al., (1994), the differences between the real period P(E) and the ephemeris period $P_e$, as well as its relative rate of change, were calculated and are presented in Figure 8, where one can notice the continuous increasing of P(E).

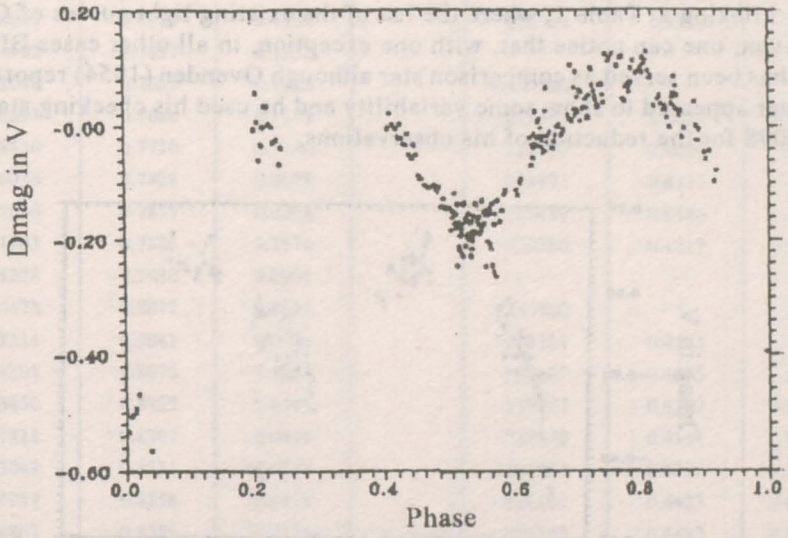



Fig. 3 – 1989 V light curve of GO Cygni, based on observations made at the Bucharest Observatory in ten nights during 1989.

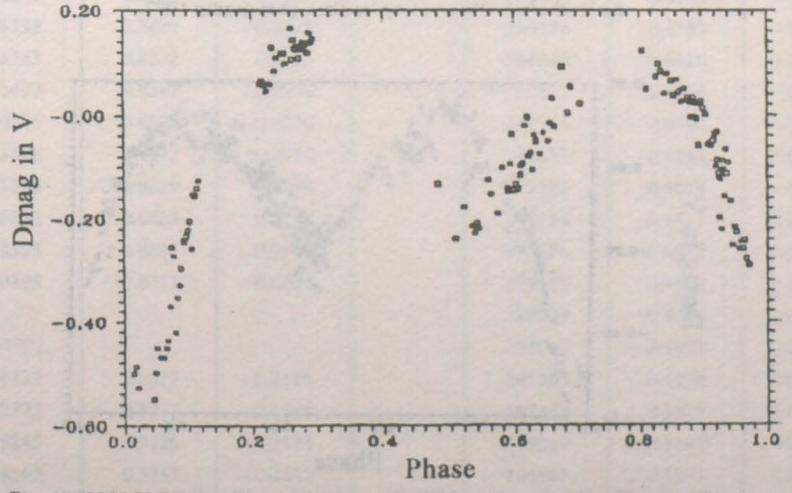



Fig. 4 – 1990 V light curve of GO Cygni, based on observations made at the Bucharest Observatory in seven nights during 1990.

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## 4. SUMMARY AND DISCUSSION

GO Cygni is a short period eclipsing binary exhibiting a β Lyrae-type light curve. According to Pearce (1933), the spectral types of its components are A9n and A0n. But, Pierce (1939) and Ovenden (1954), from their light curves solutions, noted that the secondary's spectrum might be of F5 & F8 or later, respectively; while Popper (1957) questioned if it is visible at all. Moreover, the spectroscopic mass-ratio found by Pearce (1933) differs significantly by the new one and by many photometrics.

Looking at Table 1, where the list of the existing light curves of GO Cyg are given, one can notice that, with one exception, in all other cases BD + 35° 4197 has been served as comparison star although Ovenden (1954) reported that this star appeared to show some variability and he used his checking star BD + 34° 4098 for the reduction of his observations.

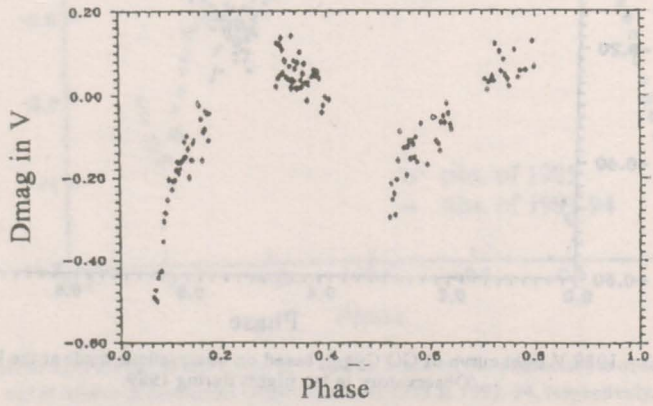



Fig. 5 – 1992 V light curve of GO Cygni, based on observations made at the Bucharest Observatory in six nights during 1992.

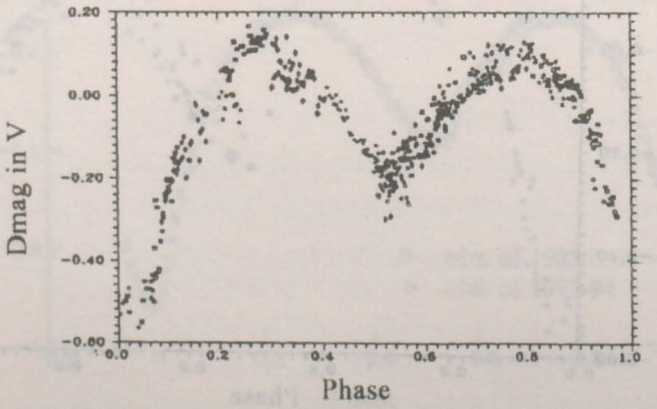



Fig. 6 – The V light curve of GO Cygni, based on all individual observations carried out at the Bucharest Observatory from 1989 to1992.

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*Table 4*

Individual Observations of GO Cygni made at Bucharest Observatory during 1989, 1990 & 1992

| Hel. JD. | Phase | Delta V |
|---|---|---|
| 2447790. | | |
| 259853 | 0.7451 | 0.1231 |
| 262978 | 0.7494 | 0.1110 |
| 265061 | 0.7524 | 0.1158 |
| 267492 | 0.7557 | 0.1233 |
| 273048 | 0.7635 | 0.1035 |
| 276693 | 0.7686 | 0.1171 |
| 278430 | 0.7710 | 0.1190 |
| 286936 | 0.7828 | 0.0673 |
| 288846 | 0.7855 | 0.0988 |
| 291103 | 0.7886 | 0.1114 |
| 294228 | 0.7930 | 0.0905 |
| 300478 | 0.8017 | 0.1021 |
| 302214 | 0.8041 | 0.0908 |
| 304298 | 0.8070 | 0.1065 |
| 316450 | 0.8239 | 0.0345 |
| 321311 | 0.8307 | 0.0466 |
| 323048 | 0.8331 | 0.0518 |
| 324957 | 0.8358 | 0.0375 |
| 326867 | 0.8385 | 0.0273 |
| 329124 | 0.8416 | 0.0335 |
| 330687 | 0.8439 | 0.0267 |
| 332249 | 0.8460 | 0.0138 |
| 334332 | 0.8489 | −0.0012 |
| 336763 | 0.8522 | 0.0392 |
| 338499 | 0.8547 | 0.0352 |
| 340756 | 0.8578 | 0.0298 |
| 343186 | 0.8612 | 0.0278 |
| 345096 | 0.8639 | 0.0190 |
| 346832 | 0.8663 | 0.0249 |
| 348395 | 0.8685 | −0.0188 |
| 350305 | 0.8711 | −0.0029 |
| | | |
| 2447792. | | |
| 234833 | 0.5017 | −0.2186 |
| 242773 | 0.5077 | −0.1788 |
| 246245 | 0.5126 | −0.2178 |
| 248502 | 0.5157 | −0.2237 |
| 252669 | 0.5215 | −0.2107 |

| Hel. JD. | Phase | Delta V |
|---|---|---|
| 254752 | 0.5244 | −0.1955 |
| 256488 | 0.5268 | −0.1994 |
| 258224 | 0.5293 | −0.2001 |
| 260134 | 0.5319 | −0.2024 |
| | | |
| 2447802. | | |
| 218589 | 0.4062 | −0.0039 |
| 221193 | 0.4098 | 0.0220 |
| 223971 | 0.4137 | 0.0034 |
| 227443 | 0.4186 | −0.0049 |
| 229700 | 0.4217 | −0.0188 |
| | | |
| 2447802. | | |
| 232131 | 0.4251 | 0.0036 |
| 233867 | 0.4275 | −0.0215 |
| 235777 | 0.4302 | −0.0315 |
| 237860 | 0.4331 | −0.0334 |
| 241853 | 0.4386 | −0.0505 |
| 244457 | 0.4423 | −0.0569 |
| 246540 | 0.4452 | −0.0325 |
| 249318 | 0.4490 | −0.0511 |
| 251288 | 0.4517 | −0.0870 |
| 252964 | 0.4541 | −0.0926 |
| 254526 | 0.4563 | −0.0935 |
| 258346 | 0.4616 | −0.1049 |
| 262513 | 0.4674 | −0.1095 |
| 267721 | 0.4747 | −0.1053 |
| 270672 | 0.4788 | −0.1135 |
| 272582 | 0.4814 | −0.1066 |
| 275186 | 0.4851 | −0.1411 |
| 277096 | 0.4877 | −0.1271 |
| 279179 | 0.4906 | −0.1440 |
| 280568 | 0.4926 | −0.1422 |
| 284047 | 0.4974 | −0.1445 |
| 285603 | 0.4996 | −0.1829 |
| 287339 | 0.5020 | −0.1499 |
| 289249 | 0.5047 | −0.1713 |
| 290985 | 0.5071 | −0.1543 |
| 292547 | 0.5093 | −0.1376 |

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*Table 4 (continued)*

| Hel. JD. | Phase | Delta V |
|---|---|---|
| 294110 | 0.5114 | −0.1698 |
| 299020 | 0.5141 | −0.1740 |
| 297061 | 0.5155 | −0.1883 |
| 298971 | 0.5182 | −0.1375 |
| 300534 | 0.5204 | −0.1694 |
| 301749 | 0.5221 | −0.1612 |
| 304353 | 0.5257 | −0.1539 |
| 306089 | 0.5281 | −0.1711 |
| 307478 | 0.5301 | −0.1682 |
| 308864 | 0.5320 | −0.1855 |
| 310603 | 0.5344 | −0.1915 |
| 312165 | 0.5366 | −0.1355 |
| 313381 | 0.5383 | −0.1380 |
| 315290 | 0.5409 | −0.1732 |
| 317026 | 0.5434 | −0.1601 |
| 318589 | 0.5455 | −0.1838 |
| 319978 | 0.5475 | −0.1310 |
| 321714 | 0.5499 | −0.1499 |
| 323797 | 0.5528 | −0.1545 |
| 325013 | 0.5545 | −0.1485 |
| 326749 | 0.5569 | −0.1560 |
| 328485 | 0.5593 | −0.1457 |
| 330395 | 0.5620 | −0.1365 |
| 332651 | 0.5651 | −0.1727 |
| 334040 | 0.5671 | −0.1539 |
| 330603 | 0.5692 | −0.1314 |
| 336818 | 0.5709 | −0.1309 |
| 338207 | 0.5746 | −0.1684 |
| 341679 | 0.5777 | −0.1485 |
| 344631 | 0.5818 | −0.1203 |
| 345846 | 0.5835 | −0.1134 |
| | | |
| 2447805. | | |
| 227476 | 0.5982 | −0.1323 |
| 229213 | 0.6007 | −0.1119 |
| 230428 | 0.6024 | −0.1211 |
| 233206 | 0.6062 | −0.0996 |
| 234768 | 0.6084 | −0.0922 |
| 245185 | 0.6229 | −0.0196 |
| 246921 | 0.6253 | −0.0676 |
| 248483 | 0.6275 | −0.0547 |
| 249872 | 0.6295 | −0.0123 |
| 251261 | 0.6314 | 0.0255 |
| 253344 | 0.6343 | −0.0201 |
| 254560 | 0.6360 | 0.0072 |
| 256122 | 0.6382 | 0.0116 |
| 257337 | 0.6399 | −0.0026 |
| 258900 | 0.6420 | −0.0238 |
| 260289 | 0.6440 | −0.0454 |
| 262372 | 0.6469 | −0.0158 |
| 263587 | 0.6484 | 0.0262 |
| 264976 | 0.6505 | 0.0659 |
| | | |
| 2447813. | | |
| 189212 | 0.6907 | 0.1052 |
| 192685 | 0.6955 | 0.0662 |
| 194421 | 0.6979 | 0.0945 |
| 196504 | 0.7008 | 0.0503 |
| 198414 | 0.7035 | 0.0605 |
| 200497 | 0.7064 | 0.0592 |
| 205879 | 0.7139 | 0.0466 |
| 207962 | 0.7168 | 0.0132 |
| 209525 | 0.7190 | 0.0529 |
| 211435 | 0.7216 | 0.1001 |
| 212476 | 0.7231 | 0.0974 |
| 213518 | 0.7245 | 0.0693 |
| 215080 | 0.7267 | 0.0595 |
| 216296 | 0.7284 | 0.0504 |
| 217685 | 0.7303 | 0.0432 |
| 218726 | 0.7318 | 0.0323 |
| 220118 | 0.7337 | 0.0343 |
| 222025 | 0.7364 | 0.0703 |
| 223587 | 0.7386 | 0.0720 |
| 224620 | 0.7400 | 0.0534 |
| 225844 | 0.7417 | 0.0401 |
| 247546 | 0.7720 | 0.0467 |
| 249976 | 0.7753 | 0.0433 |
| 251712 | 0.7778 | 0.0661 |
| 252928 | 0.7794 | 0.0706 |
| 254490 | 0.7816 | 0.0574 |
| 256053 | 0.7838 | 0.0717 |
| 257094 | 0.7853 | 0.0745 |
| 2447814. | | |
| 274920 | 0.2033 | −0.0033 |

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*Table 4 (continued)*

| Hel. JD. | Phase | Delta V |
|---|---|---|
| 277351 | 0.2067 | 0.0583 |
| 278740 | 0.2086 | 0.0091 |
| 280823 | 0.2115 | −0.0575 |
| 283427 | 0.2152 | −0.0305 |
| 285337 | 0.2178 | 0.0007 |
| 287247 | 0.2205 | 0.0364 |
| 289851 | 0.2241 | 0.0359 |
| 291760 | 0.2268 | 0.0058 |
| 294538 | 0.2306 | 0.0014 |
| 296448 | 0.2333 | −0.0262 |
| 299747 | 0.2379 | −0.0217 |
| 303045 | 0.2425 | −0.0636 |
| 304955 | 0.2452 | 0.0351 |
| | | |
| 2447821. | | |
| 304365 | 0.9969 | −0.3960 |
| 307664 | 0.0014 | −0.5333 |
| 312351 | 0.0080 | −0.5063 |
| 315303 | 0.0121 | −0.5010 |
| 317039 | 0.0145 | −0.4932 |
| 320511 | 0.0193 | −0.4690 |
| 333879 | 0.0380 | −0.5648 |
| | | |
| 2447823. | | |
| 191219 | 0.6257 | −0.0334 |
| 192955 | 0.6281 | −0.0383 |
| 194864 | 0.6307 | −0.0304 |
| 197642 | 0.6346 | −0.0328 |
| 199205 | 0.6368 | −0.0195 |
| 200767 | 0.6390 | −0.0170 |
| 202503 | 0.6414 | −0.0344 |
| 203892 | 0.6433 | −0.0232 |
| 206323 | 0.6467 | −0.0415 |
| 207885 | 0.6489 | −0.0427 |
| 209621 | 0.6513 | −0.0274 |
| 210837 | 0.6530 | −0.0173 |
| 212920 | 0.6559 | −0.0102 |
| 215177 | 0.6590 | −0.0364 |
| 216739 | 0.6612 | −0.0372 |
| 218302 | 0.6634 | −0.0258 |
| 219691 | 0.6653 | −0.0144 |
| 221253 | 0.6675 | −0.0002 |
| 223163 | 0.6702 | −0.0009 |
| | | |
| 2447823. | | |
| 224726 | 0.6723 | 0.0026 |
| 226462 | 0.6748 | 0.0153 |
| 227851 | 0.6767 | 0.0121 |
| 229066 | 0.6784 | 0.0060 |
| 234274 | 0.6856 | 0.0190 |
| 236010 | 0.6881 | 0.0126 |
| 237399 | 0.6900 | 0.0110 |
| 238962 | 0.6922 | 0.0098 |
| | | |
| 2447826. | | |
| 241392 | 0.6956 | 0.0286 |
| 242781 | 0.6975 | 0.0189 |
| 244344 | 0.6997 | 0.0277 |
| 245733 | 0.7016 | 0.0226 |
| 248684 | 0.7057 | 0.0087 |
| 251288 | 0.7093 | 0.0303 |
| 253371 | 0.7122 | 0.0330 |
| 255628 | 0.7154 | 0.0310 |
| 256844 | 0.7171 | 0.0176 |
| | | |
| 2447825. | | |
| 252548 | 0.4975 | −0.1491 |
| 256021 | 0.5024 | −0.1606 |
| 257583 | 0.5045 | −0.1182 |
| 259145 | 0.5067 | −0.1692 |
| 260882 | 0.5091 | −0.2450 |
| 262644 | 0.5113 | −0.2130 |
| 264354 | 0.5140 | −0.1936 |
| 265569 | 0.5157 | −0.1808 |
| 266958 | 0.5176 | −0.2287 |
| 268347 | 0.5195 | −0.1346 |
| 270604 | 0.5227 | −0.1655 |
| 273208 | 0.5263 | −0.1907 |
| 275465 | 0.5295 | −0.2282 |
| 276159 | 0.5304 | −0.2161 |
| 277548 | 0.5324 | −0.1875 |
| 278764 | 0.5341 | −0.1582 |
| 280152 | 0.5360 | −0.1851 |
| 283104 | 0.5401 | −0.1660 |

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*Table 4 (continued)*

| Hel. JD. | Phase | Delta V |
|---|---|---|
| 285882 | 0.5440 | −0.1889 |
| 289180 | 0.5486 | −0.1756 |
| 290916 | 0.5510 | −0.1711 |
| 292479 | 0.5532 | −0.2465 |
| 294215 | 0.5556 | −0.2028 |
| 296646 | 0.5590 | −0.2079 |
| 298034 | 0.5609 | −0.2161 |
| 299597 | 0.5631 | −0.2436 |
| 301159 | 0.5653 | −0.2554 |
| 303243 | 0.5682 | −0.2625 |
| | | |
| 2447826. | | |
| 186164 | 0.7983 | 0.0951 |
| 188594 | 0.8016 | 0.0942 |
| 190677 | 0.8045 | 0.1170 |
| 193282 | 0.8082 | 0.1129 |
| 195191 | 0.8108 | 0.1270 |
| 197101 | 0.8135 | 0.0956 |
| 199011 | 0.8162 | 0.0821 |
| 200400 | 0.8181 | 0.0870 |
| 225052 | 0.8524 | 0.0486 |
| 226441 | 0.8544 | 0.0559 |
| 227830 | 0.8563 | 0.0555 |
| 229393 | 0.8585 | 0.0688 |
| 230782 | 0.8604 | 0.0518 |
| | | |
| 2447826. | | |
| 250226 | 0.8875 | 0.0415 |
| 251441 | 0.8892 | 0.0005 |
| 253004 | 0.8914 | 0.0250 |
| 255087 | 0.8943 | −0.0064 |
| 260816 | 0.9023 | −0.0456 |
| 262900 | 0.9052 | −0.0578 |
| 266893 | 0.9107 | −0.0440 |
| 271927 | 0.9177 | −0.0972 |
| 274011 | 0.9207 | −0.0777 |
| | | |
| 2448160. | | |
| 244178 | 0.2137 | 0.0607 |
| 246609 | 0.2171 | 0.0651 |
| 248229 | 0.2194 | 0.0581 |
| 251354 | 0.2237 | 0.0468 |

| Hel. JD. | Phase | Delta V |
|---|---|---|
| 255405 | 0.2294 | 0.0595 |
| 258067 | 0.2331 | 0.1294 |
| 260440 | 0.2364 | 0.0860 |
| 263044 | 0.2400 | 0.1114 |
| 268021 | 0.2470 | 0.1203 |
| 270220 | 0.2500 | 0.1190 |
| 272303 | 0.2529 | 0.0994 |
| 279363 | 0.2628 | 0.1072 |
| 280579 | 0.2645 | 0.1445 |
| 282662 | 0.2674 | 0.1276 |
| 284340 | 0.2697 | 0.1335 |
| 286771 | 0.2731 | 0.1099 |
| 288507 | 0.2755 | 0.1270 |
| 290822 | 0.2787 | 0.1400 |
| 292847 | 0.2816 | 0.1271 |
| 294641 | 0.2841 | 0.1365 |
| 296319 | 0.2864 | 0.1197 |
| 298518 | 0.2895 | 0.1582 |
| 300197 | 0.2918 | 0.1399 |
| 302049 | 0.2944 | 0.1468 |
| | | |
| 2448176. | | |
| 219659 | 0.4711 | −0.1230 |
| 226893 | 0.4811 | −0.1330 |
| 245932 | 0.5077 | −0.2386 |
| 255713 | 0.5213 | −0.1759 |
| 264682 | 0.5338 | −0.2136 |
| 267287 | 0.5374 | −0.2267 |
| 269254 | 0.5402 | −0.2083 |
| 271511 | 0.5433 | −0.2184 |
| 282275 | 0.5583 | −0.1252 |
| 285516 | 0.5628 | −0.1512 |
| 293039 | 0.5733 | −0.1898 |
| 298016 | 0.5802 | −0.1002 |
| 304034 | 0.5886 | −0.1419 |
| 318560 | 0.6089 | −0.0980 |
| 320990 | 0.6122 | −0.0924 |
| 323768 | 0.6161 | −0.0204 |
| 326083 | 0.6193 | −0.0042 |
| | | |
| 2448176. | | |
| 332217 | 0.6279 | −0.1053 |

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*Table 4 (continued)*

| Hel. JD. | Phase | Delta V |
|---|---|---|
| 336268 | 0.6335 | −0.0521 |
| | | |
| 2448177. | | |
| 227649 | 0.8754 | 0.0095 |
| 230253 | 0.8790 | 0.0291 |
| 232337 | 0.8819 | −0.0028 |
| 234420 | 0.8848 | 0.0279 |
| 236851 | 0.8882 | −0.0032 |
| 239281 | 0.8916 | −0.0061 |
| 241191 | 0.8943 | −0.0575 |
| 259246 | 0.9194 | −0.0372 |
| 260809 | 0.9216 | −0.0987 |
| 262719 | 0.9253 | −0.0858 |
| 264281 | 0.9265 | −0.1128 |
| 266017 | 0.9289 | −0.0187 |
| 267580 | 0.9310 | −0.1050 |
| 269142 | 0.9332 | −0.1028 |
| 270878 | 0.9356 | −0.1557 |
| 272788 | 0.9383 | −0.1123 |
| 274871 | 0.9412 | −0.0902 |
| 277128 | 0.9443 | −0.1654 |
| 279212 | 0.9473 | −0.2513 |
| 281642 | 0.9506 | −0.2160 |
| 283205 | 0.9528 | −0.2257 |
| 286851 | 0.9579 | −0.2585 |
| 288587 | 0.9603 | −0.2455 |
| 290496 | 0.9630 | −0.2572 |
| 291885 | 0.9649 | −0.2413 |
| 293795 | 0.9676 | −0.2774 |
| 295705 | 0.9702 | −0.2884 |
| 297614 | 0.9729 | −0.2910 |
| | | |
| 2448180. | | |
| 194206 | 0.0085 | −0.4740 |
| 197504 | 0.0131 | −0.5001 |
| 200167 | 0.0168 | −0.4879 |
| 202366 | 0.0198 | −0.5281 |
| 219958 | 0.0444 | −0.5505 |
| 221810 | 0.0469 | −0.4990 |
| 224281 | 0.0503 | −0.4518 |
| 257747 | 0.8834 | 0.0241 |
| 260235 | 0.8869 | 0.0340 |
| 262608 | 0.8902 | 0.0244 |
| 265039 | 0.8936 | 0.0213 |
| 268222 | 0.8980 | −0.0276 |
| 270884 | 0.9017 | 0.0143 |
| 273719 | 0.9057 | 0.0032 |
| 276555 | 0.9096 | −0.0551 |
| 279738 | 0.9141 | −0.0572 |
| 282168 | 0.9175 | −0.0480 |
| 286509 | 0.9235 | −0.0330 |
| 289113 | 0.9271 | −0.1978 |
| 291428 | 0.9304 | −0.2202 |
| 293453 | 0.9332 | −0.0872 |
| 295768 | 0.9364 | −0.0651 |
| | | |
| 2448184. | | |
| 198021 | 0.5867 | −0.1061 |
| 200220 | 0.5897 | −0.1462 |
| 202477 | 0.5929 | −0.0937 |
| 204387 | 0.5955 | −0.0370 |
| 206817 | 0.5989 | −0.1435 |
| 209306 | 0.6024 | −0.1325 |
| 211505 | 0.6054 | −0.1415 |
| 214225 | 0.6092 | −0.1148 |
| 223484 | 0.6221 | −0.0777 |
| 225973 | 0.6256 | −0.0714 |
| | | |
| 2448184. | | |
| 229155 | 0.6300 | −0.0384 |
| 231702 | 0.6336 | −0.0465 |
| 236736 | 0.6404 | −0.0735 |
| 240324 | 0.6456 | −0.0343 |
| 246169 | 0.6537 | −0.0498 |
| 248658 | 0.6572 | −0.0136 |
| 251378 | 0.6610 | 0.0335 |
| 254213 | 0.6649 | 0.0216 |
| 263530 | 0.6779 | 0.0939 |
| 269491 | 0.6862 | 0.0054 |
| 272848 | 0.6909 | 0.0539 |
| 283669 | 0.7060 | 0.0228 |
| | | |
| 2448216. | | |
| 244347 | 0.2342 | 0.1473 |

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*Table 4 (continued)*

| Hel. JD. | Phase | Delta V |
|---|---|---|
| 263733 | 0.2612 | 0.1661 |
| 266511 | 0.2612 | 0.1968 |
| | | |
| 2448882. | | |
| 379042 | 0.3047 | 0.0210 |
| 384598 | 0.3124 | 0.0487 |
| 378086 | 0.3159 | 0.1104 |
| 389979 | 0.3199 | 0.0836 |
| 391658 | 0.3223 | 0.0808 |
| 397213 | 0.3300 | 0.0376 |
| 398892 | 0.3324 | 0.0161 |
| 400570 | 0.3347 | 0.0285 |
| 402827 | 0.3378 | 0.0817 |
| 404852 | 0.3407 | 0.0625 |
| 407051 | 0.3437 | −0.0130 |
| | | |
| 2448883. | | |
| 293323 | 0.5771 | −0.1174 |
| 294985 | 0.5808 | −0.0685 |
| 304650 | 0.5943 | −0.1685 |
| 312289 | 0.6049 | −0.0543 |
| 314777 | 0.6084 | −0.0540 |
| 317092 | 0.6116 | −0.1104 |
| 319696 | 0.6152 | −0.1165 |
| 321664 | 0.6180 | −0.1339 |
| 323963 | 0.6210 | −0.0654 |
| 326467 | 0.6247 | −0.0501 |
| 329129 | 0.6284 | −0.0668 |
| 331849 | 0.6322 | −0.0192 |
| 333990 | 0.6352 | −0.0431 |
| 336536 | 0.6387 | −0.0707 |
| 338620 | 0.6416 | −0.0820 |
| | | |
| 2448892. | | |
| 247232 | 0.0532 | −0.4917 |
| 254755 | 0.0637 | −0.4936 |
| 256838 | 0.0666 | −0.4739 |
| 259153 | 0.0698 | −0.5024 |
| 261005 | 0.0724 | −0.4436 |
| 262683 | 0.0747 | −0.4308 |
| 264593 | 0.0774 | −0.4278 |
| 266329 | 0.0798 | −0.4276 |
| 267949 | 0.0821 | −0.3543 |
| 269801 | 0.0846 | −0.3061 |
| 271537 | 0.0872 | −0.2853 |
| 273274 | 0.0895 | −0.2874 |
| 275299 | 0.0923 | −0.2554 |
| 277209 | 0.0950 | −0.2038 |
| 282126 | 0.1019 | −0.2303 |
| 284037 | 0.1045 | −0.2080 |
| 286005 | 0.1073 | −0.2062 |
| 287741 | 0.1097 | −0.1813 |
| 290056 | 0.1129 | −0.1613 |
| 292255 | 0.1160 | −0.1883 |
| 293933 | 0.1183 | −0.1784 |
| 295727 | 0.1208 | −0.1632 |
| 297579 | 0.1234 | −0.1561 |
| 299778 | 0.1264 | −0.1136 |
| 301167 | 0.1284 | −0.1492 |
| 302845 | 0.1307 | −0.0989 |
| 304639 | 0.1332 | −0.1312 |
| 306318 | 0.1356 | −0.2000 |
| 308690 | 0.1389 | −0.1563 |
| 310774 | 0.1419 | −0.1568 |
| 314188 | 0.1463 | −0.1097 |
| 320438 | 0.1552 | −0.0198 |
| 322290 | 0.1578 | −0.0235 |
| 324199 | 0.1605 | −0.0579 |
| 326225 | 0.1633 | −0.1577 |
| 328135 | 0.1660 | −0.0884 |
| 329929 | 0.1685 | −0.0815 |
| 331491 | 0.1706 | −0.0379 |
| 333401 | 0.1733 | −0.0430 |
| 333484 | 0.1762 | −0.1099 |
| 337336 | 0.1788 | −0.0447 |
| | | |
| 2448893. | | |
| 298641 | 0.5181 | −0.2587 |
| 300435 | 0.5206 | −0.2975 |
| 302287 | 0.5232 | −0.2156 |
| 303849 | 0.5253 | −0.2537 |
| 306511 | 0.5290 | −0.2369 |
| 308537 | 0.5319 | −0.2911 |
| 310331 | 0.5344 | −0.2029 |

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*Table 4 (continued)*

| Hel. JD. | Phase | Delta V |
|---|---|---|
| 312588 | 0.5375 | – 0.1578 |
| 314497 | 0.5402 | – 0.0872 |
| 320863 | 0.5490 | – 0.1274 |
| 323699 | 0.5530 | – 0.1928 |
| 325551 | 0.5556 | – 0.1231 |
| 327518 | 0.5583 | – 0.1109 |
| 329023 | 0.5604 | – 0.1523 |
| 331280 | 0.5636 | – 0.1099 |
| 332900 | 0.5658 | – 0.1195 |
| 2448893. | | |
| 334636 | 0.5682 | – 0.1494 |
| 336199 | 0.5704 | – 0.1019 |
| 338398 | 0.5735 | – 0.1607 |
| 2448936. | | |
| 209358 | 0.3020 | 0.1348 |
| 211615 | 0.3052 | 0.1244 |
| 213872 | 0.3083 | 0.0416 |
| 215955 | 0.3112 | 0.1201 |
| 218154 | 0.3143 | 0.1363 |
| 220838 | 0.3172 | 0.0547 |
| 222263 | 0.3200 | 0.0831 |
| 225272 | 0.3242 | 0.0427 |
| 227414 | 0.3272 | 0.0382 |
| 229323 | 0.3299 | 0.0689 |
| 231985 | 0.3336 | 0.1404 |
| 233606 | 0.3358 | 0.0391 |
| 235978 | 0.3391 | 0.0352 |
| 237807 | 0.3417 | 0.0171 |
| 239972 | 0.3447 | 0.0766 |
| 241592 | 0.3470 | 0.0152 |
| 243386 | 0.3495 | 0.1033 |
| 244891 | 0.3515 | 0.0362 |
| 246164 | 0.3533 | 0.0181 |
| 247495 | 0.3552 | 0.0521 |
| 249173 | 0.3575 | 0.0738 |
| 251256 | 0.3604 | 0.0297 |
| 253108 | 0.3630 | – 0.0122 |
| 255191 | 0.3659 | 0.0242 |
| 254439 | 0.3649 | 0.0308 |
| 258953 | 0.3711 | 0.0442 |
| 260342 | 0.3731 | 0.0469 |
| 262252 | 0.3757 | 0.0600 |
| 263930 | 0.3781 | 0.0481 |
| 265203 | 0.3798 | 0.0477 |
| 267344 | 0.3828 | 0.0568 |
| 269138 | 0.3853 | 0.0403 |
| 272959 | 0.3907 | – 0.0419 |
| 274578 | 0.3929 | – 0.0232 |
| 276604 | 0.3957 | – 0.0043 |
| 282159 | 0.4035 | – 0.0110 |
| 2448937. | | |
| 217283 | 0.7063 | 0.0396 |
| 220697 | 0.7111 | 0.0119 |
| 222665 | 0.7138 | 0.0307 |
| 225732 | 0.7181 | 0.0354 |
| 227410 | 0.7204 | 0.0462 |
| 229378 | 0.7232 | 0.0339 |
| 231229 | 0.7257 | 0.1234 |
| 237306 | 0.7342 | 0.1048 |
| 238926 | 0.7365 | 0.0572 |
| 242340 | 0.7412 | 0.0843 |
| 244366 | 0.7440 | 0.0477 |
| 248301 | 0.7495 | 0.0384 |
| 249748 | 0.7515 | 0.0212 |
| 257387 | 0.7622 | 0.1105 |
| 262364 | 0.7691 | 0.0432 |
| 264447 | 0.7720 | 0.0590 |
| 273822 | 0.7831 | 0.0600 |
| 278567 | 0.7917 | 0.2327 |
| 281114 | 0.7952 | 0.1291 |
| 283660 | 0.7988 | 0.0675 |

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*Table 5*

Coefficients of second and third order approximation of the (O–C) diagram of GO Cygni

| | Second | Third |
|---|---|---|
| $a_0$ | – 0.001471 | 0.002639 |
| $a_1$ | – 0.032667 | – 0.038842 |
| $a_2$ | 0.202446 | 0.086650 |
| $a_3$ | | 0.214041 |
| rms | 0.0025 | 0.0022 |
| c | 37000.00 | 37000.00 |

Taking seriously enough Ovenden's (1954) suggestion, we choose another star as comparison in our first observations; but, in all other cases we did as all the other observers. Now, comparing the mean light curves coming out from our observations of 1989, 1990 & 1992 with those of 1993–94, we can see that there is a shift. This is presented in Figure 9, and being of the order of 0.082 in V light, is very probably due to a possible variability of BD + 35° 4197, as Ovenden suggested, but it could also be owned to intrinsic variations of the system itself. This must be checked with new observations of the system and of BD + 35° 4197.

Comparing the mean light curves of our "first" and "last" observations, obtained at Athens in 1985 and Bucharest in 1993–1994, respectively, one can see that the depths of the primary minima coincide; but, the secondary are deeper and wider in Bucharest curves. Moreover, the latter at phase 0.2 presents a small hump. In V filter the above mentioned features are less evident than in B. Taking into account that the observations were performed in periods differing by 8 years, those features could be due to the activity of the system.

The curve observed in 1989–90 & 92 shows much more fluctuations. Moreover, Maximum I is obviously higher than Maximum II and the curve

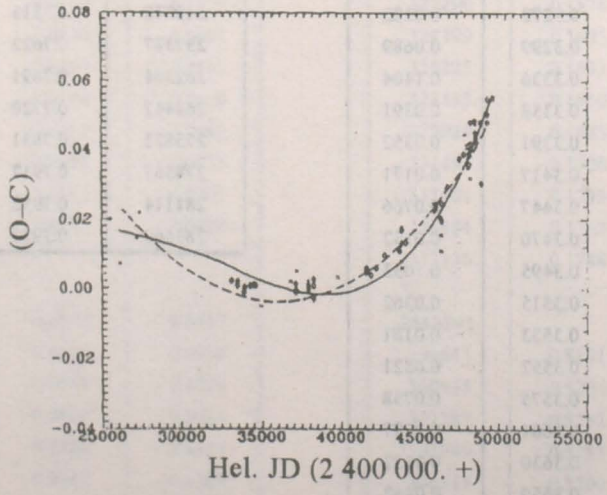



Fig. 7. – The (O–C) diagram of GO Cygni, based on 121 minima times and its description by a quadratic (dashed line) and a third (continous line) polynominal.

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presents a hump near the phase 0.2. This feature is also present in the 1993–94 curve. Another feature, present in both 1989–90 & 92 and 1993–94 light curves of GO Cygni is a small flat zone at phase 0.1. Finally, one can see that in both cases the descending branch of primary minimum is smoother than the ascending one. This may be owned to a perturbation caused by a stream of matter.

From Tables 2 & 3, where the results of all light curves analyses of GO Cyg are given, one can see the differences in the geometric and photometric elements of the system. Especially large differences are found in the value of the inclination of the orbit. Moreover, from the geometric elements derived from our study of GO Cyg, one can get an idea about the degree of contact of the two components of the system; and since its light curves are of b Lyrae-type, GO Cygni belongs to the so-called near contact systems (e.g. Shaw 1993).

Moreover, combining the mean geometric elements of GO Cyg derived from the analysis of our data and the spectroscopic ones the absolute elements

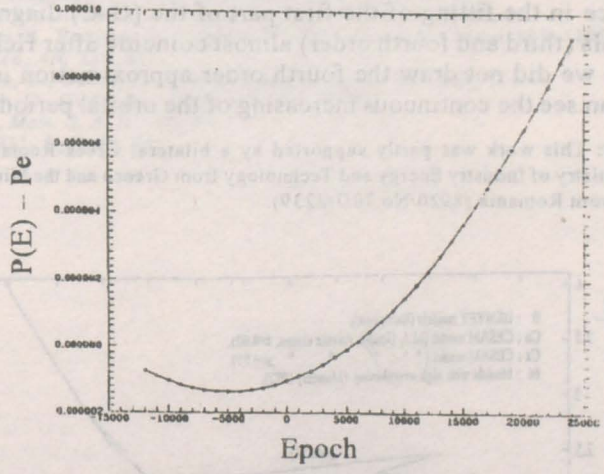



Fig. 8 – The differences between the real period P(E) and that of the ephemeris Pe.

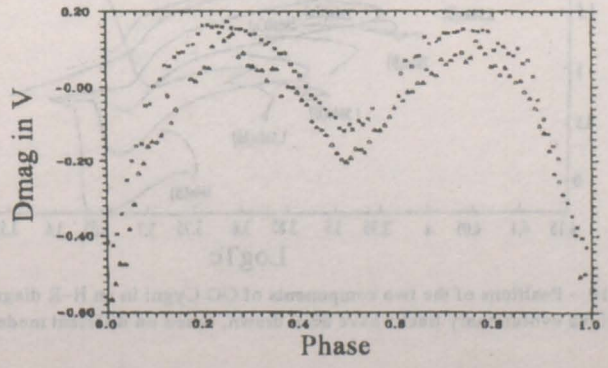



Fig. 9 – The difference of the mean light curves of 1989–90 (^) & 92 (^) and those of 1993–94 (*).

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of the system can be found. From these, we can get an idea about its evolutionary status, although there are not new values for the velocities $K_1$ & $K_2$ and we shall be limited to the very old ones. So, in Figure 10 where the position of the two components of GO Cyg in an H–R diagram is given together with some evolutionary tracks, based on different models, both components of GO Cygni are very close to the ZAMS. Similar result we get if the two components are in the mass-radius (M–R) or in the mass-luminosity (M–L) diagrams of Hilditch et al. (1988).

As regards the orbital period of GO Cyg, one can see its variation from both the (O–C) and the P(E)–$P_e$ diagrams of the system – presented in Figures 7 and 8 – and as well as from the various proposed ephemeris formulae. Moreover, from Table 5, where the coefficients of the different order approximations are given, one can see that the third order is the most appropriate one.

But, since the rms error of the fourth differs so little to that of the third, a fourth order approximation is also pretty good. In fact doing so, we found a small difference in the fitting of the first part of the (O–C) diagram, while the two polynomials (third and fourth order) almost coincide after Hel. JD. 243400; for this reason we did not draw the fourth order approximation in Figure 7. In Figure 8 one can see the continuous increasing of the orbital period of GO Cygni.

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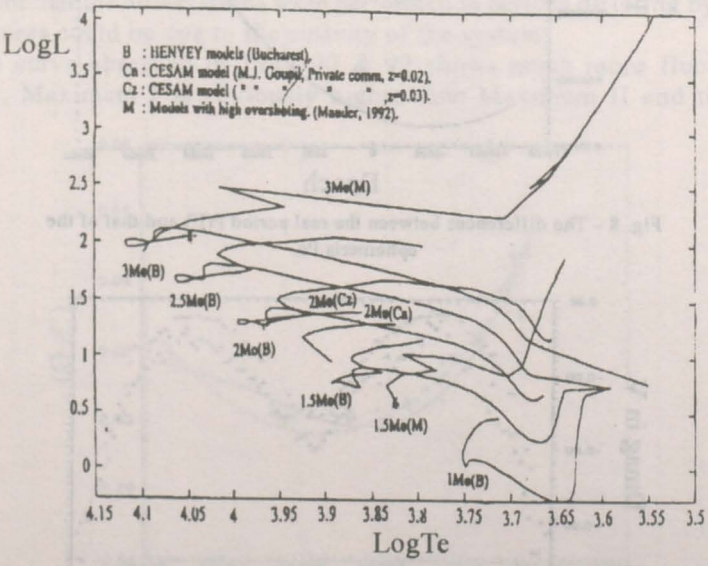



Fig. 10 – Positions of the two components of GO Cygni in an H–R diagram. Various evolutionary tracks have been drawn, based on different models.

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# MULTIPERIODIC BEHAVIOUR OF RT PERSEI

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*Abstract.* The period variation of RT Persei is resumed. A new, semi-empirical ephemeris, consisting in a polynomial term and a "multiperiodic" term, is established on the basis of 577 individual times of primary minimum light, covering 98 years. The frequencies detected in the amplitude spectrum of the $O - C$ curve are believed to characterize the physical mechanisms occurring in the system.

*Key words:* variable stars, eclipsing binary systems, Algol binaries

## 1. INTRODUCTION

Since its discovery, in 1904, the eclipsing binary system RT Persei enjoys a constant interest from both observational and theoretical viewpoint. Such an interest is especially determined by the period variability emphasized by the corresponding $O - C$ diagram.

In order to describe the shape of the $O - C$ curve, different models were suggested. Thus, some authors presumed the existence of one (e.g. Friboes-Conde & Herczeg 1973, Todoran 1979, 1980; Rafert 1982) or more (Panchatsaram 1981; see also Abhyankar & Panchatsaram 1984) unseen companions. It is well known that the consequence of such a dynamical situation on the minimum light times is a modulation with a periodic function due to the orbital motion (Kopal 1978, p.304). In the more complex case of a multiple stellar system (under the hypothesis of a hierarchical dynamical system) (see Roy 1982, p.155, and Abhyankar & Panchatsaram 1984) one may consider, in a first approximation at least, a multiperiodic modulating function. Note that even if one considers the hypothesis of a third body only, we must take into account the existence of the perturbations induced by the hypothetical companion in the orbital motion of the close pair (Kopal 1978, p.276). Since we have no *a priori* informations about the actual situation therefrom, we cannot estimate their order of magnitude, too.

Another possible mechanism considered in order to explain the shape of the $O - C$ curve is the mass transfer (e.g. Mancuso *et al.* 1977). Thus, they found the variability in the level of maxima of about 0.035 mag, humps appearing in the descending and ascending branches of both primary and secondary minima, as well as variability in the height of maxima and in the depth of minima. The conclusion of the authors' analysis is that the cause of these phenomena belongs intrinsically



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to the system. They also suggest an explanation: the ejection of particles along retrograde orbits. Todoran (1980) postulated two sudden changes of the orbital period which he considered as effects of mass transfer. Sanwal & Chaubey (1981), explained the distortion of the light curve near the shoulders of the primary minimum on the basis of a disk around the brighter component. Chaubey (1984) discusses the observed trend of period variation between 1895 and 1967 in terms of a non-conservation of orbital angular momentum.

On the other hand, having in view the spectral type of the secondary component, which lies between G5IV–G8IV (Mancuso *et al.* 1977) and K0:IV (Svechnikov 1986), we cannot dismiss the possibility of a period modulation in connection with magnetic activity cycles (e.g. Hall 1989, 1990; Applegate 1992). At the same time we note that, until now, it is not clear whether the variability of light curve between primary and secondary minima (e.g. Mancuso *et al.* 1977, Sanwal & Chaubey 1981) is caused by the pulsation of primary (as in the case of other Algols such as AB Cas or Y Cam) or by the starspot activity of the cool secondary related to the magnetic activity cycle.

We can conclude that owing to the above mentioned diversity of hypotheses invoked in order to explain the behaviour of the $O-C$ curve, as well as their nature, we must take into account the possibility of the coexistence of some mechanisms too (e.g. mass transfer + activity cycles, mass transfer + companion(s), magnetic activity cycles + companion(s), or, why not ?, magnetic activity cycles + mass transfer + companion(s)). Such a working hypothesis must be retrieved in the mathematical expression of an ephemeris able to describe the run of the $O-C$ curve.

The aim of the present paper is to establish a new ephemeris for primary minimum light times of RT Persei on the basis of a mathematical model fulfilling the above mentioned condition. This means it must have two components: one able to describe the possible secular trend, of unknown shape, and the other adequate for describing the observed alternating behaviour. The most straightforward way for solving this problem is the use of a model constituted from a polynomial term plus a "multiperiodic" one, based on the superposition of a given number of trigonometric polynomials (Pop 1996). Of course, such an approach involves the assumption of additivity – in a first approximation, at least – of the corresponding effects.

With that end in view we shall use 577 times of primary minimum light (527 – visual, 39 – photographic, 11 – photoelectric) extending over a time base of 98 years (1898–1996). These have been collected in the ***BAV Database for Times of Minima of Eclipsing Binaries*** and may be sent by F. Agerer, at request.

## 2. THE METHOD

The methodology used here in order to establish a more refined ephemeris for the period variation of RT Persei was already described in a previous paper (Pop 1996). It briefly consists in testing polynomial trends of various degrees. After

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their subtraction from the observed $O-C$ curve, the amplitude spectrum (Deeming 1975, Kurtz 1985) of the corresponding residuals is computed, and the frequencies of the highest peak(s) are retained. The final step is to test different combinations of polynomial degrees, number of frequencies, and number of harmonics for each frequency in connection with the least squares fitting of the following ephemeris

$$t^c_{n_j} = t_0 + \sum_{k=1}^{K} \tau_k \, n_j^k + \sum_{l=1}^{L} \sum_{m=1}^{M_l} \tau_{lm} \sin\left(\Omega_{lm} \, n_j + \varphi_{lm}\right), \qquad (1)$$

where, obviously, $\tau_1$ is the orbital period, $\Omega_{lm} = m\,\Omega_l$, $\Omega_l = 2\pi f_{0l}$, and $f_{ol} = f_l \tau_1$. After a preliminary fitting of this model, all parameters occurring above are improved using the differential correction (DC) method.

The final parameter set for the multiperiodic ephemeris is selected so that the standard deviation of the corresponding residuals is minimum. In the DC improving process we are looking after the fulfilling of the stationarity of the standard deviation of the residuals, and that of the model parameters between two consecutive steps. We consider – in a first approximation, at least – that the achievement of these criteria as well as the convergence of improving process might be an indication of the agreement between the model and the reality, in connection with the period modulation mechanisms discussed in the first section.

All the computations involved by the above described methodology have been performed by means of the computer programs written by one of the authors (A.P.).

## 3. RESULTS AND CONCLUDING REMARKS

The $O-C$ curve (see Fig.1) has been computed using the following preliminary linear ephemeris

$$t^c_{n_j} = 2433616.4314 + 0.849405 \cdot n_j. \qquad (2)$$

In order to remove the secular trend we determined the following parabolic ephemeris

$$\begin{aligned} t^c_{n_j} &= 2433616.41926 + 0.849404390 \cdot n_j - 1.218 \cdot 10^{-10}. \\ &\pm 0.00085 \quad \pm 0.000000044 \quad \pm 0.042 \cdot 10^{-10} \end{aligned} \qquad (3)$$

We retained the following three frequencies $f_{01} = 5.70 \cdot 10^{-5}$, $f_{02} = 9.07 \cdot 10^{-5}$, and $f_{03} = 14.65 \cdot 10^{-5}$ from the amplitude spectrum of the residuals resulted from the subtraction of the parabolic secular trend (see Fig.2). We recall that these frequencies are dimensionless because time is expressed by means of the orbital cycles number elapsed since $t_0$. The final results of least squares fitting and DC

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improvement of the ephemeris (1) are displayed in Table 1, where amplitudes are given in days, and phases in radians. The values of the periods corresponding to the three frequencies are 42.97 yr, 26.53 yr, and 19.98 yr respectively.

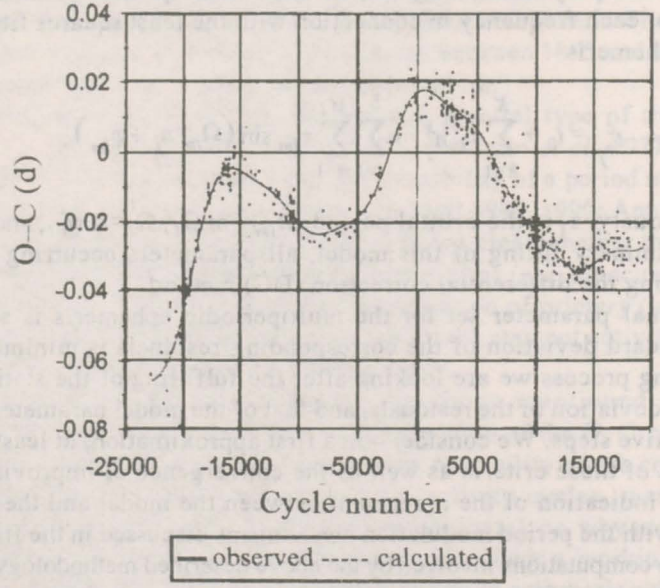



Fig. 1. – Theoretical O–C curve.

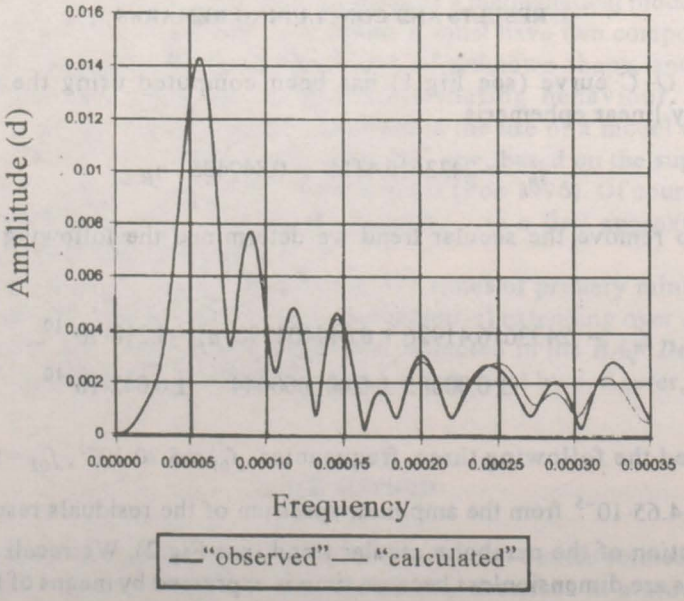



Fig. 2. – Amplitude spectrum of synthetic data.

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*Table 1*

| Polynomial term | Multiperiodic term | | | | | |
|---|---|---|---|---|---|---|
| | $f_{o1} = 5.412 \cdot 10^{-5} \pm 0.032 \cdot 10^{-5}$ | | $f_{o1} = 8.766 \cdot 10^{-5} \pm 0.143 \cdot 10^{-5}$ | | $f_{o1} = 11.642 \cdot 10^{-5} \pm 0.570 \cdot 10^{-5}$ | |
| $t_0 = 2433616.41612 \pm 0.00045$ | $\tau_{11} = 0.01771$ | $\varphi_{11} = 0.759$ | $\tau_{21} = 0.00384$ | $\varphi_{21} = 2.45$ | $\tau_{31} = 0.0059$ | $\varphi_{31} = 2.34$ |
| | $\pm 0.00041$ | $\pm 0.026$ | $\pm 0.00083$ | $\pm 0.14$ | $\pm 0.0041$ | $\pm 0.18$ |
| $\tau_1 = 0.849404835 \pm 0.000000028$ | $\tau_{12} = 0.0064$ | $\varphi_{12} = 5.73$ | $\tau_{22} = 0.00172$ | $\varphi_{22} = 2.09$ | – | |
| | $\pm 0.0048$ | $\pm 0.28$ | $\pm 0.00032$ | $\pm 0.21$ | | |
| $\tau_2 = -1.056 \cdot 10^{-5} \pm 0.026 \cdot 10^{-5}$ | – | | – | | – | |

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Figure 1 also contains the theoretical $O - C$ curve computed using the ephemeris (1), while Figure 2 displays the amplitude spectrum of synthetic data generated on the basis of the same ephemeris. Naturally, the sampling of these data (from the viewpoint of the corresponding cycle numbers) was identical to that of the original ones. One easily observes the good resemblance of the two spectra at low frequencies. The discrepancies which can be seen especially at higher frequencies are due to the noise associated with the timing errors as well as with other more subtle causes probably related to real or apparent (spurious) variations of the minimum light profile. The standard deviation of the final residuals is about 7.05 minutes. Such a quite large value is obviously due to the visual observations which are predominant.

Concerning the parameters of the semi-empirical ephemeris, some specifications are needed. The parabolic term, describing the secular period variation of RT Per, can be interpreted in terms of mass transfer (see references quoted in Sect.1). The significance of the three periodicities which seems to be involved in the period variability of RT Per is more intricate, and therefore we do not approach it here. Future investigations are necessary in order to establish their nature (... and actuality!) as well as the possible relationships between them. In this context, the frequencies given in Table 1 will constitute observational constraints for testing concrete mechanisms for period modulation. As a consequence, in the future, a particular importance must be given to eclipses timing; searching for eventual chromospheric CaII, H and K emission lines as well as full orbit Hα spectroscopy may provide necessary informations about the physical mechanisms occurring in RT Per.

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# THE USE OF CHARACTERISTIC TIME TO THE STUDY OF PERIOD VARIABILITY PHENOMENA

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*Abstract.* The notion of characteristic time associated with a variable star with changing period is resumed. A possibility to use it in order to classify period variation phenomena is described. The practical aspects of the application in concrete observational situations are approached by taking into account an ephemeris for light time extrema with high degree of generality. The observation planning is also considered from the viewpoint of the characteristic time.

*Key words:* variable stars, stellar photometry, periodicity analysis, data analysis

## 1. INTRODUCTION

The variable star researcher encounters a complex situation due to the increasing amount of moments of minimum or maximum light times, and, on the other hand, the relatively low level of the methods intended to their analysis (i.e. the corresponding $O-C$ curves) (Kalimeris *et al.* 1994). Of course, we compare them with the methods for light curve analysis. At the same time, both observational and theoretical research pointed out new physical mechanisms which can explain the complex behaviour of the variable stars' $O-C$ curves. This context requires a revaluation of the attitude with respect to this problem. The above mentioned paper constitutes an important step in this sense.

In a previous paper (Pop 1994), developing an idea of Fullerton (1986), we approached the problem of variable stars' period variability from the viewpoint of its notional frame. There, we considered that the actual period $P_p$ of a variable star may be considered as being the result of the superposition of a main (unperturbed) constant period $P_p^0$ and a perturbation due to one or more causes. We defined (with slightly modified notations) the associated characteristic time as follows

$$\tau = \max\left\{\Delta_j t \,\middle\|\, \Delta_n P_p \middle| < \varepsilon_t,\ \Delta t < P_p^0/2,\ n - n_1 \geqslant 2\right\},\ j = 2,\ldots,N_0, \tag{1.1}$$

where

$$P_p(n) = T_n - T_{n-1} \tag{1.2}$$

is the instantaneous period,

$$\Delta_n P_p = P_p(n) - P_p(n_1) \tag{1.3}$$



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represents the period change between the cycles $n_1$ and $n(\equiv n_F)$, $\varepsilon_t$ is the accuracy of the observation time determination, $\Delta_j t = t_j^n - t_1^{n_1}$ is the time interval elapsed between the instants $t_1^{n_1}$ *and* $t_j^n$ (the subscript $j$ being the observation count number, and the subscript $n$ being the number of the corresponding variation cycle), $\Delta t = t_j^n - t_{j-1}^n$ is the sampling interval, and $N_0$ the number of observations on the variable star light curve.

Thus, given an equally spaced observational data set on a variable star with changing period, the associated characteristic time is the largest time interval, measured since the time of the first observation, for which the following conditions are fulfilled: *(i)* the period change between the cycles corresponding to the first and the last observation is smaller than the timing accuracy, *(ii)* the sampling interval is shorter than an unperturbed half-period, and *(iii)* the observations cover at least two light variation cycles.

In our above quoted paper we also illustrated the notion of characteristic time on the basis of two simple cases of period variation laws: a polynomial one, and an arbitrary periodic one. We have to mention that the result corresponding to the second law was given erroneously (see Eq.(13) in Pop (1994)). The correct form of this equation is

$$\Delta_k P_p = \left| \sum_{m=1}^{M} 2\Pi_m \cos\left[\frac{\Omega_m}{2}(k+1)\right] \sin\left[\frac{\Omega_m}{2}(k-1)+\Phi_m\right] \right| \leqslant \left[ \sum_{m=1}^{M} 2\Pi_m \right].$$

Note that the characteristic time as defined above is a quantity which features the perception of a given period variability phenomenon through a given observation and measurement system (telescope + filter + radiation detector + data acquisition system). In other words, it quantifies the observer's ability to get informations about the period variability of a given variable star. We also can say that the characteristic time is a subjective quantity measuring some kind of observer's temporal inertia with respect to the given period variability phenomenon.

In the present paper, whose aim is to continue our first contribution on this subject, we shall consider some theoretical and practical aspects related to the characteristic time, in order to favour its use in connection with concrete observational situations.

## 2. A CLASSIFICATION OF PERIOD VARIABILITY PHENOMENA

In this section we shall approach the problem of the classification of period variability phenomena of variable stars by using the notion of characteristic time. Let us denote $\xi(n) = \left|\Delta_n P_p\right|$, and $\eta(n) = \left|\mathrm{d}(\Delta_n P_p)/\mathrm{d}n\right|$. Also, let $n_\varepsilon$ that value of

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$n$ for which the condition $\xi(n_\varepsilon) < \xi(n_\varepsilon + 1) \geqslant \varepsilon_t$ is fulfilled, and $n_0$ be the cycle number corresponding to the first zero of $\eta(n)$, considered since the initial one $n_I$, i.e. $\eta(n) = 0$.

With these defintions, we can distinguish the following three distinct cases:

**2.1.** $\xi(n_I + 1) > \varepsilon_t$ and $n_I + 1 < n_0$. This situation features a very rapid variability phenomenon of the principal period. There is no characteristic time in the sense of our definition (see also the condition *(iii)* therein);

**2.2.** $n_I < n_\varepsilon < n_0$ and $\xi(n_0) > \varepsilon_t$. Here it is possible to define a characteristic time. We simply have $\tau = (n_\varepsilon - n_I) P_p^0$;

**2.3.** $\xi(n_0) < \varepsilon_t$ or even $\xi(n_0) \ll \varepsilon_t$. This case corresponds to a slow or even very slow period variability phenomenon – let us call *period microvariability*. It is possible to compute an inferior limit of the characteristic time $\tau_{min} = (n_0 - n_I) P_p^0$. For $n > n_0$, it is necessary to check the validity of the used ephemeris (see Sections 3 and 4 below) for the observations in the interval $n_I$, $n_I + 1$, ..., $n_F = n_0$ of length $\tau_{min}$; then the study will be resumed.

The above classification can be illustrated by the following situation. Consider a variable star with changing period observed simultaneously visually and photoelectrically (possibly using a high speed photoelectric photometer). Depending on the shape of period variation, the same light curve segment could reveal, e.g., a *period microvariability* phenomenon for the visual observer, and a rapid period variability for the photoelectric one.

The practicalities concerning the above considerations will be discussed in Section 4, on the basis of a concrete model for the period variation.

## 3. MODELLING THE *O–C* CURVE

The following approach of the problem of mathematical description of the *O–C* curve relies on the already known possibility of coexistence of several mechanisms yielding period variations. Thus (referring to different types of variables, such as eclipsing close binary systems, Cepheids, RR Lyrae stars, δ Scuti stars), for example, secular period changes may be due to mass transfer, mass loss, or structural evolutive changes (e.g. Huang 1956, Huang 1963, Szabados 1983, Rodríguez *et al.* 1995), while alternating period changes may be caused both by the presence of at least one unseen companion (e.g. Kopal 1978, Abhyankar & Panchatsaram 1984), by apsidal motion (e.g. Gimenez & Garcia-Pelayo 1983)), by mass transfer (e.g. Biermann & Hall 1973), by Blazhko effect (e.g. Szeidl 1988), and by magnetic activity cycles (e.g. Hall 1989, Applegate 1992). Period changes may also be caused by motions in the stellar core (van't Veer 1986). Further complications may occur under the influence of stellar surface details, which could alter the shape of light curve's minima (e.g. by causing asymmetries), during their own evolution or during orbital motion; the effect of these distortions is that of a spurious period variation (e.g. van't Veer 1973).

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Let us suppose that we are disposing of $N$ observed extremum (minima or maxima) light times on a given variable star, which shows a complex period variability phenomenon $\left\{T^0_{n_j}\right\}_{j=1,\ldots,N}$, where $n_j$ refers to the respective variation cycle. As we know, the $O-C$ curve may be drawn by computing the corresponding residuals with respect to a given preliminary linear ephemeris

$$T^c_{n_j} = \widetilde{T}_0 + \widetilde{P}_p\, n_j \,, \tag{3.1}$$

and then

$$(O-C)^o_{n_j} = T^o_{n_j} - T^c_{n_j} \,, \tag{3.2}$$

where $\widetilde{T}_0$ and $\widetilde{P}_p$ are respectively the initial epoch and the variation principal period. In order to describe the behavior of the $O-C$ curve, we shall assume a model based on a linear combination of known functions. Such a model with a high degree of generality is that obtained by the superposition of a polynomial (at least as a first approximation) trend, and a "multiperiodic (in general) term" based on the superposition of a given number of trigonometric polynomials. Thus, one easily can see that the proposed model makes no ***a priori*** assumption about the mechanism(s) of the period change displayed by the $O-C$ curve. However, its mathematical expression is able to describe – even in an empirical way – the effect of many real mechanisms acting in stars or stellar systems.

As in practice, we shall consider the lack of any information about the corresponding frequencies, amplitudes and phases. The first step in the analysis of the $O-C$ curve is obviously to remove the above mentioned trend. This may be done by least squares fitting of the following polynomial model

$$(O-C)^{c_1}_{n_j} = \Delta T_0 + \Delta P_p + \sum_{k=2}^{K} \tau_k n_j^k \,, \tag{3.3}$$

and then, by computing the new residuals

$$(O-C)^o_{n_j} - (O-C)^{c_1}_{n_j} \,, \tag{3.4}$$

which clearly contain the "multiperiodic term" only. Now, we can approach the problem of periodicity detection in the time series corresponding to (3.4). The most adequate tool for this aim is the discrete Fourier transform (Deeming 1975, Kurtz 1985). Due to the unavoidable presence of additive noise in the $O-C$ curve because of the uncertainties in the time determinations, the use of Lomb-Scargle periodogram (Lomb 1976, Scargle 1982, Horne & Baliunas 1986) may be recommendable. It could be necessary to perform several such analyses in order to test different values for the polynomial degree taking into account the existing informations about the physical conditions in the respective variable star (e.g. mass transfer).

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Let $\Omega_1, \Omega_2, \ldots, \Omega_L$ be the "secondary" angular frequencies found in the respective power spectrum. We note that they are dimensionless, since in (3.4) (via (3.3)) time occurs through the cycle number corresponding to the unperturbed principal period $P_p^0$

$$\Omega_l = 2\pi f_{0l} = 2\pi f_{sl} / f_p^0 = 2\pi P_p^0 / P_{sl}, \quad l = 1,\ldots,L. \tag{3.5}$$

We also assume that for any $l$ fundamental frequency there are $M_l - 1$ detectable harmonics. At this stage we have at our disposal informations enough in order to tackle the so-called estimation problem. Having in view the above mentioned features for the period variability phenomenon, let us write its analytical expression

$$(O-C)^{c_2}_{n_j} = \Delta T_0 + \Delta P_p n_j + \sum_{k=2}^{K} \tau_k n_j^k + \sum_{l=1}^{L}\sum_{m=1}^{M_l} \tau_{lm} \sin(\Omega_{lm} n_j + \varphi_{lm}), \tag{3.6}$$

where

$$\Omega_{lm} = m\Omega_l. \tag{3.7}$$

The numerical values of the model's unknown parameters $\Delta T_0, \Delta P_p, \{\tau_k\}_{k=2,\ldots,K}$, $\{\tau_{lm}, \varphi_{lm}\}_{l=1,\ldots,L; m=1,\ldots,M_L}$ can be determined by least squares fitting of the curve ruled by Eq.(3.6) to the initial $O$–$C$ curve (Eq.(3.2)).

Finally, we can write the new ephemeris which is able to describe the time repartition of the extremum light times

$$T^c_{n_j} = T_0 + \sum_{k=1}^{K} \tau_k n_j^k + \sum_{l=1}^{L}\sum_{m=1}^{M_l} \tau_{lm} \sin(\Omega_{lm} n_j + \varphi_{lm}), \tag{3.8}$$

in which we obviously have

$$\tau_1 = P_p. \tag{3.9}$$

Of course, if necessary (and our own experience proved that it generally does!) all parameters in Eq.(3.8) (including this time $\{\Omega_{lm}\}_{l=1,\ldots,L; m=1,\ldots,M_l}$) may be improved by using the differential corrections method.

## 4. CHARACTERISTIC TIME AND OBSERVATIONS' PLANNING

Having in view the new ephemeris (3.8), we shall use it in order to approach the problem of observations' planning by means of the previously defined notion of characteristic time.

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First of all, let us determine the instantaneous period expression (1.2) (e.g. Kopal 1978), or, in other words, the analytic expression for the period modulation law. Using Eq.(3.8) we immediately get

$$P_p(n) = \sum_{k=1}^{K} \tau_k \left( n^k - (n-1)^k \right) + \sum_{l=1}^{L} \sum_{m=1}^{M_l} \Pi_{lm} \cos(\Omega_{lm} n + \Psi_{lm}) \qquad (4.1)$$

in which

$$\Pi_{lm} = 2\sin(\Omega_{lm} / 2)\tau_{lm},$$
$$\Psi_{lm} = \varphi_{lm} - \Omega_{lm} / 2 \qquad (4.2)$$

The next step is to deduce, for the given initial epoch $T_0$, the law ruling the period variations between the cycles $n_1$ and $n = n_F$, where $n_1$ corresponds to the last observed extremum light time, and $n_F$ must be determined taking into account the remarks in Section 2 related to the characteristic time. Thus, using Eq.(1.3) and (3.8), we obtain

$$\Delta_n P_p = -P_p(n_1) + \sum_{k=1}^{K} \tau_k \left( n^k - (n-1)^k \right) + \sum_{l=1}^{L} \sum_{m=1}^{M_l} \Pi_{lm} \cos(\Omega_{lm} n + \Psi_{lm}). \qquad (4.3)$$

Differentiating $\Delta_n P_p$ with respect to $n$, i.e. with respect to time, we get

$$\frac{d(\Delta_n P_p)}{dn} = \sum_{k=1}^{K} k \tau_k \left( n^{k-1} - (n-1)^{k-1} \right) - \sum_{l=1}^{L} \sum_{m=1}^{M_l} \Omega_{lm} \Pi_{lm} \sin(\Omega_{lm} n + \Psi_{lm}). \qquad (4.4)$$

Now, having at our disposal all the quantities occurring in Sect.2, it is possible to evaluate them in practice on the basis of the given set of model parameters corresponding to the given $O-C$ curve. Thus, according to the concrete shape of period changes (mainly in the cases featured by a complicated behaviour), the period of one and the same variable star may be perceived at different time intervals as being rapid variable (case (2.1)), or very slow variable, even non-variable.

The above proposed approach may also be used in order to establish the best observing seasons for a given variable star, particularly when its period proved to have an intricate behaviour. In such a case one may be interested in obtaining phase diagrams as smooth as possible. This means that changes in principal period must be very low, i.e. the value of $\Delta_n P_p$ associated to the corresponding time interval must be close to zero (when sudden period changes are lacking). Such seasons can easily be identified with characteristic times whose lengths are determined by the actual particular period modulation law.

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# GENERAL FORMULATION OF SOME PROBLEMS OF CELESTIAL MECHANICS FOR CONSTRUCTING SOLUTIONS USING COMPUTERS

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*Abstract.* Some problems of celestial mechanics are exposed in a modern way, the characteristic of the motion theory being not the order (power of a small parameter), but an *a priori* given accuracy (even if the theory accuracy depends implicitly on the small parameter). The mathematical formulation is given for six special problems, whose solutions can be obtained with the aid of the "Bogolyubov processor".

*Key words:* celestial mechanics – theory of motion – symbolic processors

## 1. INTRODUCTION

To formulate certain research problems in celestial mechanics, that in our opinion have some topicalness today, it is necessary to take as a basis one of the definitions given to the celestial mechanics as branch of the modern science. There are some definitions for the celestial mechanics, the first one being proposed by Laplace (to whom the name of this science belongs).

In the sequel we shall limit ourselves to present Laplace's definition generalized by Duboshin (1963) and the (concise) definition given by Grebenikov and Ryabov (1966).

According to Duboshin (1963), "the celestial mechanics is that part of astronomy which deals with the study of translation, rotation, and deformation motions of any heavenly bodies, natural or artificial, under the action of attractive or repulsive forces, resistance forces due to the presence of dust, gaseous, or liquid media, electromagnetic forces, light pressure forces, etc." The author emphasizes the fact that all objects studied by astronomy belong to the framework of celestial mechanics, and the difference between this latter one and the other branches of astronomy consists mainly of the research methods.

The definition given by Grebenikov and Ryabov (1966) is: "The celestial mechanics is that science which, based on the fundamental laws of mechanics, and using mathematical methods, studies the motions of all heavenly bodies, whatever they are."

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Comparing the classical and modern definitions, one notices that the main difference is determined by the advances of science during the cosmic era. On the one hand, there exist today many artificial bodies; this means that one has the possibility not only to investigate the dynamics of celestial bodies, but also to design the orbits for certain such objects. On the other hand, the astonishing advances in computer science gave rise to new technologies in all branches of modern science – in celestial mechanics, too. Within this context, some classical but always actual problems of celestial mechanics can be reformulated in a modern way.

## 2. THE PROBLEMS

The equations which describe the motion of celestial bodies are nonlinear, hence their solutions cannot be obtained in closed form; see the theorems of Bruns (1884) and Poincaré (1892–1899). In general these solutions are determined by means of numerical, or iterative analytical methods (or both). In their turn, the iterative methods used formerly either asymptotic series (in Poincaré's meaning) or successive approximations. Besides these ones, the canonical (Hamiltonian) dynamics uses successfully (due to Poincaré, too) methods based on transformations of canonical variables. Due to the enormous intricacy of the analytical expressions, the above mentioned analytical methods were used only in a first – seldom in a second – approximation. The determination of the third approximation was practically impossible. After all, until the middle of our century we know only Spiru Haret's famous thesis, in which he proved the existence of purely secular terms of third order in the analytic expressions for the major axes of planetary orbits. Haret's (1878) result was confirmed by Meffroy (1958) (see also Pal 1991; Schneider 1993).

This classical problem (that of the planetary motion in the Solar System) can be presented in a modern form as follows:

PROBLEM 1. *Determination of the solutions of the differential equations which describe the motions of the planets in the Solar System, by means of classical methods, but with an a priori given accuracy.*

To formulate this problem means to construct the respective asymptotic or numerical-analytical theory with a requested accuracy. This means in its turn that the parameter of this theory is not the number of successive approximations, although the theory accuracy naturally depends on this parameter (but not implicitly). This is the difference between the two formulations, classical (theory of first, second order, etc.) and modern (theory with an a priori given accuracy).

The practical solution of this problem became possible only recently, due to the modern computer techniques.

Within the framework of analytical methods of celestial mechanics, to investigate the motion of celestial bodies, one of the authors of this paper (EG) proposed a new method, which uses asymptotic series and is based on the averaging principle of the respective differential equations (Grebenikov and Ryabov 1983;

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Grebenikov 1986, 1994). The perturbations are determined in analytic form, with any required accuracy, as a result of the integration of a system of quasilinear PDEs. We called this method "generalization of the Bogolyubov-Krylov method". It is a well-known fact that Bogolyubov created a very efficient method for the study of oscillations as solutions of ODE systems called "systems of differential equations in standard form". The equations of celestial mechanics are much more complex, and Bogolyubov's method cannot be applied to this case. That is why EG proposed a generalization of this method, which allows the use of Cauchy's well-known method for ODEs in celestial mechanics (see e.g. Kamke 1959). Within the framework of this method, the perturbations of any order are solutions of the system

$$\left(\partial u/\partial y,\omega(x)\right)=F(x,y)\equiv\sum_{\|k\|\geq 1}F_k(x)e^{i(x,y)}, \tag{1}$$

where $x$, $u$, $F$ are $m$-dimensional vectors; $y$, $\omega$ $(x)$, $k$ are $n$-dimensional vectors; the series is an $n$-multiple Fourier series; the right-hand side expression is the product between the $m\times n$ Jacobi matrix and $\omega$ $(x)$; and

$$\|k\|=|k_1|+|k_2|+\ldots+|k_n|;\quad i=\sqrt{-1}$$
$$k_s=\pm 1,\pm 2,\ldots;\quad s=1,2,\ldots,n. \tag{2}$$

In the monographs of Grebenikov and Ryabov there is proved the fact that the exact general solution of (1) can be presented through generalized multiple Fourier series with variable divisors in the form

$$u(x,y)=\sum_{\|k\|\geq 1}{}'\frac{f_k(x)}{i\left(k,\omega(x)\right)}e^{i(k,y)}+\varphi(x), \tag{3}$$

where $(k,\omega\ (x))$ is a nonzero scalar product, while $\varphi(x)$ is an arbitrary vector function. When $(k,\omega\ (x))=0$, the solution is

$$u(x,y)=\frac{1}{n}\sum_{s=1}^{h}\frac{y_s}{\omega_{s(x)}}\sum_{\|k\|}{}'F_{k(x)}e^{i(k,y)}+\sum_{\|k\|}{}''\frac{F_{k(x)}}{i\left(k,\omega(x)\right)}e^{i(k,y)}+\varphi(x). \tag{4}$$

In the sum $\sum''$ the components of the vector index $k$ have only values for which $\left(k,\omega(x)\right)\neq 0$, whereas in the sum $\sum''$ the vector index $k$ has values for which $(k,\omega\ (x))=0$.

PROBLEM 2. *Using equations* (1) *and solutions* (3) *and* (4), *one constructs the theory of motion of celestial bodies whose frequencies* (*or mean motions*) *are rationally commensurable* (*the frequency resonance does exist*) *of the type*

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$$\left(k,\omega\left(x^*\right)\right)=0, \qquad \|k\|\neq 0, \tag{5}$$

*where $x^*$ is the set of points in the phase space $(x,y)$ for which the frequencies are resonant.*

Problems 1 and 2 are traditionally formulated, but to solve them we need a symbolic software which allows operations with analytical expressions of the type

$$\sum_s \sum_k \frac{A_k x_1^{s_1} \ldots x_m^{s_m}}{\left(k,\omega(x)\right)} e^{i(k,y)}; \qquad s=(s_1,\ldots,s_m). \tag{6}$$

We shall call series (6) "Bogolyubov processor" and formulate the following problem:

PROBLEM 3. *Using one of the strong symbolic computation systems* (REDUCE, MATHEMATICA, MACSYMA, MAPLE, etc.) *to create a special software for analytic operations on the "Bogolyubov processor", the constructive investigations of Problems 1 and 2 become feasible.*

PROBLEM 4. *To resume Spiru Haret's theory on the basis of the solutions of Problems 1–3 (actually the construction of the proper theory of motion "starting from Haret's formulae", as Tisserand required).*

The problem also has an important astronomical application (initiated by Spiru Haret): the explanation of the secular acceleration of the Moon's mean motion, the O–C difference of 11" being not satisfactorily explained (Haret 1937; Pal 1991).

PROBLEM 5. *To resume* (on the basis of Problems 2 and 3) *the analytical theories of interpolation for the minor planets* (1) *Ceres and* (5) *Astraea, constructed by Yarov-Yarovoj* (1954a,b, 1960), *respectively Pal* (1959, 1963), *by means of averaging schemes based on the so-called "empiric integrals", following Mojseev's* (1954a,b, 1960) *ideas.*

Although these two theories proved to be useful for the practical astronomy, the mathematical substantiation of interpolation averaging schemes in Bogolyubov's meaning was done only recently (Grebenicov and Pal 1995).

PROBLEM 6. *Construction of analytical intermediate orbits intended to represent the artificial Earth satellite motion* (using the solutions of Problems 2 and 3).

The detailed formulation of this problem can be found in the papers of Pal (1973) and Pal and Pârv (1995).

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# MODIFIED LAGRANGE-BÜRMANN SERIES FOR REPRESENTING KEPLER'S EQUATION SOLUTION

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*Abstract.* New series representing the solution of Kepler's transcendent equation are being obtained. Their convergence domain is larger than the convergence domain of the well-known Lagrange series for the same solution.

*Key words*: celestial mechanics – Kepler's equation

The prominent part of Kepler's equation

$$z = M + e \sin z \tag{1}$$

($e$ = eccentricity of the planetary orbit; $M$, z = mean and eccentric anomalies of the planet, respectively) in the theoretical astronomy is well known. Considering $z$ as unknown, and $M$ and $e$ as known quantities, equation (1) is transcendent with respect to $z$, so that the finding of the explicit function

$$z = z\,(M, e) \tag{2}$$

represents a nontrivial problem.

The first basic result in this direction consisted of the representation of the solution of equation (1) by means of Lagrange's classical series (see e.g. Hurwitz and Courant 1968; Subbotin 1968)

$$z = M + \sum_{s=1}^{\infty} \frac{e^s}{s!} \left[ \frac{\mathrm{d}^{s-1}(\sin^s z)}{\mathrm{d}z^{s-1}} \right]_{z=M} . \tag{3}$$

Series (3) actually represents the power series expansion of the function $z = z\,(g)$, the inverse of

$$g(z) = \frac{z - M}{\sin z}, \; z(0) = M, \tag{4}$$

in the point $g = e$.



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On the frontier of the disk of convergence of the inverse function $z = z(g)$ expansion in power series of $g$

$$z = M + \sum_{s=1}^{\infty} \frac{g^s}{s!} \left[ \frac{\mathrm{d}^{s-1}(\sin^s z)}{\mathrm{d}z^{s-1}} \right]_{z=M} . \tag{5}$$

must lie at least one critical point $g^*$ determinable from the condition (e.g. Evgrafov 1971)

$$g^* = g(z^*), \ (\partial g / \partial z)_{z=z^*} = 0. \tag{6}$$

In this way, for the convergence of the series (5) in the point $g = e$ (hence for the convergence of the series (3)), the condition $|g^*| > e$ must be necessarily fulfilled. By (4) and (6) this condition reads

$$\left| 1/\cos z^* \right| > 1,$$

where $z^*$ is the solution of the equation

$$\sin z^* - (z^* - M)\cos z^* = 0 \tag{7}$$

To concretize, we shall limit to the case $M = \pi/2$. One knows (see Subbotin 1968) that for this value of $M$

$$z^* = \pi/2 - ir^*, \ r^* \simeq 1.19, \tag{8}$$

and the series (3) is convergent for $|e| < |g^*| < 0.6627$.

Consider now the class $S$ of functions such that any function $\tilde{g}(z) \in S$ may be represented in the form

$$\tilde{g}(z) = \frac{z - M}{a^{-1}(\mathrm{e}\sin z + M - z) + z - M}, \tag{9}$$

where $a \in \mathbb{C}$ is a constant, and $\tilde{g}(M) = 0$.

If $\tilde{g} \in S$, then in the point which represents the solution of equation (1), $z = \bar{z}$, we shall have $\tilde{g}(\bar{z}) = 1$. Observe that

$$\tilde{g} = \frac{ag}{\mathrm{e} + (a-1)g}, \tag{10}$$

i.e. the functions $\tilde{g} \in S$ perform a homographic transformation of the plane $g$.

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The series which represents the function $z = z(\tilde{g})$ in a neighbourhood of $\tilde{g}(M) = 0$ is of the form

$$z = M + \sum_{s=1}^{\infty} \sum_{m=0}^{s-1} \frac{\tilde{g}^s}{s!} \frac{C_{s-1}^m C_s^m m!}{a^{s-m}} (1 - a^{-1})^m e^{s-m} \times \left[ \frac{d^{s-m-1}(\sin^{s-m} z)}{dz^{s-m-1}} \right]_{z=M} . \quad (11)$$

On the frontier of the disk of convergence for (11) must lie at least one critical point $\tilde{g}^*$ which fulfils the condition $\partial \tilde{g}^* / \partial z = 0$. Consequently, $\tilde{g}^* = g(z^*)$, where $z^*$ is the solution of the equation

$$\sin z^* - (z^* - M) \cos z^* = 0. \quad (12)$$

Observe that the images of the critical point for all functions $\tilde{g} \in S$ and for the function $g(z)$ are determined from the same condition (12).

To make series (11) provide a representation for the solution of equation (1), it must converge in the point $\tilde{g} = 1$. That is why we must necessarily have the condition: from $|\tilde{g}^*| > 1$ follows

$$\left| \frac{a(z^* - M)}{e \sin z^* + (a-1)(z^* - M)} \right| > 1, \quad (13)$$

from which, taking into account (12), one obtains

$$\left| 1 - a^{-1}(1 - e \cos z^*) \right| < 1. \quad (14)$$

In the case $M = \pi/2$, observing (8), we get the condition

$$\left| 1 - a^{-1}(1 - ie \cos r^*) \right| < 1. \quad (15)$$

To concretize, let us consider that $e > 0$, $e \in \mathbb{R}$; then we have $\arg(1 - ie \cos r^*) \in (-\pi/2, 0)$. From simple geometric considerations one sees that for any $e > 0$ it follows $a > 0$, and condition (15) holds. In other words, the modified Lagrange-Bürmann (hereafter MLB) series converges for $e > 0.6627$, too. This statement can be proved to be true also for any $M \in [0, \pi]$.

The above considerations can be formulated as:

THEOREM 1. *For every $e > 0$ ($e \in \mathbb{R}$) there exists $a(e) > 0$ such that the MLB series converges in the point $\tilde{g} = 1$ to the exact solution $z = \bar{z}$ of equation* (1).

*Remark.* For $a = 1$ the MLB series reduces to the Lagrange series (3) in the point $z = \bar{z}$. Actually a more general theorem holds, which we state – for sake of brevity – without proof.

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THEOREM 2. *Consider for* $\{z_0, e\} \in \mathbb{C}$ *that the equation* $z = z_0 + e\,\varphi(z)$, *where* $\varphi(z)$ *is an analytic function of* $z$ *and* $\varphi'(z) \neq 0$, *has the solution* $\bar{z}$. *Then there exists* $a(e) \in \mathbb{C}$, *depending only on* $e$, *such that the MLB series* (11) *converges to* $\bar{z}$.

To end, we must emphasize the fact that the idea of the MLB series method is exceedingly simple. We construct the expansion of the inverse function with respect to the powers of the function $\tilde{g} \in S$, which is equal to 1 in the point representing the solution of equation (1). The class $S$ is constructed in such a way that the images of the critical points $\tilde{g}^*$ be identical for any $\tilde{g}^* \in S$.

By virtue of the fact that the functions $\tilde{g}$ perform a homographic transformation of the function $g$ used in the Lagrange series, one succeeds – by means of a parameter $a$, which determines the explicit form of $\tilde{g}$ – the shift of the critical points in the plane $\tilde{g}$, obtaining the fulfilment of the convergence condition of the MLB series in the point $z = \bar{z}$.

One must emphasize the fact that the possibilities of using the MLB series method were considerably enlarged by the application of the symbolic calculation systems on today's computers.

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# ON THE SIMULTANEOUS TOTAL COLLISION FOR PARTICLE SYSTEMS WITH QUASIHOMOGENEOUS POTENTIALS

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*Abstract.* The $n$-body problem in an attracting field featured by a quasihomogeneous potential is considered. Based on an equality concerning the moment of inertia and on the Lagrange-Jacobi relation transposed for this case, one proves the impossibility of the simultaneous total collision in infinite time.

*Key words:* celestial mechanics, $n$-body problem, quasihomogeneous potentials

Consider the masses $m_i > 0 \, (i = \overline{1,n})$ in the Euclidean space $\mathbb{R}^3$, let $r_i = (x_i, y_i, z_i) \in \mathbb{R}^3$ be their position vectors, and let $r = (r_1, r_2, ..., r_n) \in \mathbb{R}^{3n}$ be the configuration of the system. Consider the motion of these masses in a force field characterized by a quasihomogeneous potential function $W = U + V$, $W, U, V: \mathbb{R}^{3n} \setminus \Delta \to [0, \infty)$, where

$$U = \sum^{*} A_{ij} / r_{ij}^{\alpha}, \quad V = \sum^{*} B_{ij} / r_{ij}^{\beta}, \quad 0 < \alpha \leqslant \beta, \tag{1}$$

$\Delta = \bigcup_{1 \leqslant i < j \leqslant n} \{ r \mid r_i = r_j \}$ stands for the set of collisions, $r_{ij} = | r_i - r_j |$, $A_{i,j}, B_{i,j} : \mathbb{R}^2 \to [0, \infty)$, $A_{ij} = A(m_i, m_j) = A(m_j, m_i) = A_{ji}$ (and a similar relation holds for $B_{ij}$), and we use the symbol $\sum^{*}$ for $\sum_{1 \leqslant i < j \leqslant n}$. (We shall also use $\sum_i = \sum_{i=1}^{n}$ and $u^2 = | u |^2$, $u \in \mathbb{R}^3$.)

The quasihomogeneous potentials constitute a natural extension of the homogeneous ones ($\alpha = \beta$). The homogeneous models (from $\alpha = \beta = 1$, the Newtonian case, up to $\alpha = \beta = 6$ or 8, the Van der Waals case) are well known in physics, and their study by the methods of mechanics led to interesting results (see Moser 1975; McGehee 1981; Diacu 1990). The attraction described by a quasihomogeneous potential also models various situations: for $\alpha = 1$, $\beta = 2$, the Maneff field (Maneff 1924, 1925, 1930a,b; Diacu 1993; Diacu et al. 1995; Mioc and Stoica 1995a,b,c), Maneff-type fields (Delgado et al. 1996), or that of Fock (see Mioc 1994) truncated



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to significant terms; for $\alpha = 1, \beta = 3$, Schwarzschild's field or the attraction of an oblate body on a test particle moving in its equatorial plane; and so forth.

Diacu (1996) tackled the three-body problem for $1 \le \alpha < \beta$, studying the collisional and near-collisional motion. In this note we shall extend some results (known for the homogeneous case) concerning the $n$-body problem to the quasihomogeneous case, leading to the theorem which proves the impossibility of simultaneous total collision in infinite time.

The equations of motion will have the form

$$m_i \ddot{r}_i = \partial W / \partial r_i = -\sum\nolimits^{*} (r_i - r_j)(\alpha A_{ij} / r_{ij}^{\alpha+2} + \beta B_{ij} / r_{ij}^{\beta+2}). \tag{2}$$

Standard results of the theory of differential equations ensure, for given initial conditions $(r,\dot{r})(0)$, the existence and uniqueness of an analytic solution of equations (2) defined on an interval $(t^-, t^+)$, $t^- < 0 < t^+$. Equations (2) being time-reversible, we may confine our study to $[0, t^+)$. The solution can be analytically extended to a maximal interval $[0, t^+)$, $0 < t^+ \le t^* \le \infty$; it is regular for $t^* = \infty$, and encounters a singularity in the opposite case.

It is easy to establish that equations (2) admit ten first integrals: those of the mass center

$$\sum_i m_i \dot{r}_i = a, \qquad \sum_i m_i r_i - t \sum_i m_i \dot{r}_i = b, \qquad a, b \in \mathbb{R}^3; \tag{3}$$

those of the angular momentum

$$\sum_i m_i r_i \times \dot{r}_i = C = constant, \qquad C \in \mathbb{R}^3; \tag{4}$$

and that of the energy

$$T - W = h = constant, \qquad h \in \mathbb{R}, \tag{5}$$

where $T = \sum_i m_i \dot{r}_i^2 / 2$ is the kinetic energy. Fixing the origin of coordinates in the mass center, (2), (4), (5) will keep their form, but (3) will become

$$\sum_i m_i r_i = 0, \qquad \sum_i m_i \dot{r}_i = 0. \tag{6}$$

We first shall establish some results analogous to the Newtonian case. Let

$$2J = \sum_i m_i r_i^2 \tag{7}$$

be the moment of inertia, and denote $M = \sum_i m_i$. Starting from the expansion of the expression $\sum_i m_i (r_i - r_j)^2$, using (6) and (7), and following exactly the argument exposed in Wintner's (1941) textbook, we obtain

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$$\sum{}^{*} m_i m_j r_{ij}^2 = 2MJ. \tag{8}$$

*Remark* 1. Two other equalities concerning the same problem, namely

$$\sum{}^{*} m_i m_j (\dot{r}_i - \dot{r}_j)^2 = 2MT,$$

$$\sum{}^{*} m_i m_j (r_i - r_j) \times (\dot{r}_i - \dot{r}_j) = MC,$$

can be easily established (see Wintner 1941, because while proving them one does not use the expression of the potential). Since these equalities will not be used in the present note, we only mention them.

Now, differentiating (7) twice with respect to time, we get

$$\ddot{J} = \sum_i m_i (\dot{r}_i^2 + r_i \cdot \ddot{r}_i). \tag{9}$$

By (1), it is easy to prove that, $\sum_i r_i (\partial W / \partial r_i) = -\alpha U - \beta V$ which, taking into account (2), gives

$$\sum_i m_i r_i \cdot \ddot{r}_i = -\alpha U - \beta V.$$

On the other hand, (5) leads to

$$\sum_i m_i \dot{r}_i^2 = 2(U + V + h).$$

Replacing these two relations in (9), this one becomes

$$\ddot{J} = (2-\alpha)U + (2-\beta)V + 2h, \tag{10}$$

which is nothing but the Lagrange-Jacobi relation transposed in the case of quasihomogeneous potentials.

*Remark* 2. An inequality analogous to that of Sundman (for the Newtonian case), namely

$$C^2 \leqslant 2J(\ddot{J} + \alpha U + \beta V) - \dot{J}^2,$$

can also be established, but we shall not use it for the purposes of this note.

We tackle now the simultaneous total collision. This means that there exists an instant $t^*$ such that $r_{ij} \to 0$, $\forall i, j$, $1 \leqslant i < j \leqslant n$, for $t \to t^*$. Taking into account formulae (7) and (8), the necessary and sufficient condition for the simultaneous total collision is $J = J(r(t)) \to 0$ when $t \to t^*$. By (1), one sees that

$$J \to 0 \Rightarrow U \to \infty,\ V \to \infty. \tag{11}$$

All that being established, one can state the following

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THEOREM. *There is no solution of equations* (2) *which leads to a simultaneous total collision for* $t \to \infty$.

*Proof.* Suppose that there exists a solution of equations (2) leading to the simultaneous total collision for $t \to \infty$. We shall consider all possible cases.

(i) If $\alpha < \beta < 2$, by (10) and (11) we have $\ddot{J} \to \infty$ for $t \to \infty$. Then there exist an instant $t_1 \geqslant 0$ and a constant $K_1 > 0$ such that $\ddot{J} > K_1$, $\forall\, t \geqslant t_1$. Integrating this inequality twice between $t_1$ and $t$, one obtains

$$J > K_1 t^2/2 + (\dot{J}_1 - K_1 t_1)t + K_1 t_1^2/2 + J_1 - \dot{J}_1 t_1, \qquad \forall t \geqslant t_1,$$

where $(J_s, \dot{J}_s) = (J, \dot{J})(r(t_s))$, $s = \overline{0,2}$, $t_0 = 0$ (see also below). It follows that $J \to \infty$ for $t \to \infty$, a contradiction.

(ii) If $\alpha = \beta < 2$, by (10), $\ddot{J} = (2-\alpha)(U+V) + 2h$, then, by (11), $\ddot{J} \to \infty$ for $t \to \infty$; the same argument and the same contradiction as in case (i).

(iii) If $\alpha < \beta = 2$, by (10), $\ddot{J} = (2-\alpha)U + 2h$, then, by (11), $\ddot{J} \to \infty$ for $t \to \infty$; the same argument and the same contradiction as in case (i).

(iv) If $\alpha = \beta = 2$, by (10), $\ddot{J} = 2h$, $\forall\, t \geqslant 0$. Integrating this relation twice between 0 and $t$, one gets

$$J = ht^2 + \dot{J}_0 t + J_0, \qquad \forall t \geqslant 0. \tag{12}$$

If $h \neq 0$, by (12), $|J| \to \infty$ for $t \to \infty$, a contradiction. If $h = 0$ and $\dot{J}_0 \neq 0$, by (12), $|J| \to \infty$ for $t \to \infty$, a contradiction. If $h = 0$ and $\dot{J}_0 = 0$, then, by (12), $J = J_0 =$ $= constant$. But, by (7), $J_0 > 0$, hence $J_0 = constant > 0$, therefore $J$ cannot tend to 0 when $t \to \infty$, a contradiction.

(v) If $\alpha < 2$, $\beta > 2$, by (1) and (10) one has

$$\ddot{J} = \sum\nolimits^{*} \left[ (2-\alpha) A_{ij} - (\beta - 2) B_{ij} / r_{ij}^{\beta-\alpha} \right] / r_{ij}^{\alpha} + h.$$

We see that $\ddot{J} \to -\infty$ for $t \to \infty$. Then there exist an instant $t_2 \geqslant 0$ and a constant $K_2 < 0$ such that $\ddot{J} < K_2$, $\forall t \geqslant t_2$. Integrating this inequality twice between $t_2$ and $t$, one obtains

$$J < K_2 t^2/2 + (\dot{J}_2 - K_2 t_2)t + K_2 t_2^2/2 + J_2 - \dot{J}_2 t_2, \quad \forall t \geqslant t_2.$$

It follows that $J \to -\infty$ for $t \to \infty$, a contradiction.

(vi) If $2 = \alpha < \beta$, by (10), $\ddot{J} = (2-\beta)V + 2h$, then, by (11), $\ddot{J} \to -\infty$ for $t \to \infty$; the same argument and the same contradiction as in case (v).

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(vii) If $2 < \alpha = \beta$, by (10), $\ddot{J} = (2-\alpha)(U+V) + 2h$, then, by (11), $\ddot{J} \to -\infty$ for $t \to \infty$; the same argument and the same contradiction as in case (v).

(viii) If $2 < \alpha < \beta$, by (10) and (11) we have $\ddot{J} \to -\infty$ for $t \to \infty$; the same argument and the same contradiction as in case (v).

All possible cases being examined, the theorem is proved.

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# SMALL STOCHASTIC PERTURBATIONS OF INTEGRABLE CANONICAL SYSTEMS

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*Abstract.* We consider the integrable canonical systems perturbed by a Brownian motion and we present an asymptotic estimation (in probability and in uniform norm) of the difference between the coordinates, the momenta and their small stochastic perturbations. The typical examples of application of these estimates are the exactly harmonic systems.

*Key words*: canonical systems – Brownian perturbations – celestial mechanics

## 1. NOTATIONS AND HYPOTHESES

We consider an $n$ points system in $\mathbf{R}^k$ determined by its coordinates $\mathbf{q} = \mathbf{q}(t) \in \in C^1([0,\infty); D_q)$, by its momenta $\mathbf{p} = \mathbf{p}(t) \in C^1([0,\infty); D_q)$ and with the initial data $(\mathbf{q}_0, \mathbf{p}_0) \in D_q \times D_p$, where $D_q, D_p \subseteq \mathbf{R}^{kn}$ are open sets. The physical motion of the $n$ points system is defined by the Hamiltonian $\mathscr{H} = \mathscr{H}(\mathbf{q}, \mathbf{p}) \in C^1(D_q \times D_p; \mathbf{R})$, and described by the following canonical system on $D_q \times D_p$ (see e. g. Arnold, 1976):

$$\begin{cases} \dot{\mathbf{q}} = \partial_2 \mathscr{H} \\ \dot{\mathbf{p}} = \partial_1 \mathscr{H}, \end{cases} \tag{1}$$

where $\partial_1 \mathscr{H}$ and $\partial_2 \mathscr{H}$ denote the partial derivatives of $\mathscr{H}$ with respect to variables $\mathbf{q}$ and $\mathbf{p}$. We shall work under the following hypotheses:

($A_1$) There exists a complete integral $-ht + S(\mathbf{q})$ with $S \in C^2(D_q; \mathbf{R})$ for the system (1), solution of the associated Hamilton-Jacobi equation

$$\mathscr{H}(\mathbf{q}, \nabla S(\mathbf{q})) = h,$$

where $h \in \mathbf{R}$ is an integration constant.

As a consequence of ($A_1$), it follows that system (1) with the initial data $\mathbf{q}(0) = \mathbf{q}_0$, $\mathbf{p}(0) = \mathbf{p}_0 = \nabla S(\mathbf{q}_0)$, is integrable (see also Arnold, 1976), with the unique solution $(\mathbf{q}, \nabla S(\mathbf{q}))$.

In this paper we shall consider the system (1) stochastically perturbed in the following sense: let $\mathbf{B}(t,\omega) : [0, \infty) \times \Omega \to \mathbf{R}^{kn}$ be a Brownian motion process



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on the probability field $(\Omega, \mathcal{K}, P^B)$; the probability $P^B\{\cdot\}$ is the law of the Brownian paths on the measurable space $(\Omega, \mathcal{K})$ (see e. g. Ikeda and Watanabe, 1989). For $\varepsilon > 0$, our aim is to define the stochastically perturbed coordinates $\mathbf{q}^\varepsilon(t, \omega): [0, \infty) \times \Omega \to D_q$ and the stochastically perturbed momentum $\mathbf{p}^\varepsilon(t, \omega): [0, \infty) \times \Omega \to D_p$, as applications (at least) continuous in $t$ and measurable in $\omega$. For this reason, let us consider the formal solution of the following stochastic differential system

$$\begin{cases} d\mathbf{q}^\varepsilon(t,\omega) = \partial_2 \mathcal{H}\left(\mathbf{q}^\varepsilon(t,\omega), \mathbf{p}^\varepsilon(t,\omega)\right)dt + \varepsilon\partial_2 \tilde{\mathcal{H}}\left(\mathbf{q}^\varepsilon(t,\omega), \mathbf{p}^\varepsilon(t,\omega)\right)d\mathbf{B}(t,\omega), \\ d\mathbf{p}^\varepsilon(t,\omega) = -\partial_1 \mathcal{H}\left(\mathbf{q}^\varepsilon(t,\omega), \mathbf{p}^\varepsilon(t,\omega)\right)dt - \varepsilon\partial_1 \tilde{\mathcal{H}}\left(\mathbf{q}^\varepsilon(t,\omega), \mathbf{p}^\varepsilon(t,\omega)\right)d\mathbf{B}(t,\omega) \end{cases} \quad (2)$$

where $\tilde{\mathcal{H}} \in C^1(D_q \times D_p; \mathbf{R})$. The differentials with respect to the Brownian motion process in (2) are in Itô's sense; they are well-defined if (see Ikeda and Watanabe, 1989):

$(A_2)$ The stochastic processes $\partial_i \mathcal{H}\left(\mathbf{q}^\varepsilon(t;), \mathbf{p}^\varepsilon(t;)\right), \partial_i \tilde{\mathcal{H}}\left(\mathbf{q}^\varepsilon(t;), \mathbf{p}^\varepsilon(t;)\right)$, $i = 1, 2$, are square integrable and adapted with respect to the Brownian filtration on $(\Omega, \mathcal{K}, P^B)$.

We shall also make the following assumption.

$(A_3)$ The stochastic system (2) with the *deterministic* initial data $(\mathbf{q}_0, \mathbf{p}_0) \in D_q \times D_p$ has a unique ($P^B$ – almost surely) pathwise globally defined solution with values in $D_q \times D_p$. In addition, the paths $\mathbf{q}^\varepsilon(\cdot,\omega)$, $\mathbf{p}^\varepsilon(\cdot,\omega)$ are continuous for $P^B$ – almost all $\omega \in \Omega$.

*Remark and example.* – Let us notice that $(A_3)$ is satisfied e. g. if $\partial_i \mathcal{H}(\mathbf{q}^\varepsilon(.,.))$, $i = 1, 2$ are locally Lipschitzian on $D_q \times D_p$, $\partial_i \tilde{\mathcal{H}}(\cdot,\cdot)$, $i = 1, 2$ are $\neq 0$ and of $C^1$ class on $D_q \times D_p$, and the above partial derivatives (for both $\mathcal{H}$ and $\tilde{\mathcal{H}}$) admit sublinear growth (see e. g. Ikeda and Watanabe, 1989).

A perturbing Hamiltonian $\tilde{\mathcal{H}}$ for the system (2) is proposed by Bismut (1981), Chapter 7 (as the solution of a certain stochastic optimization problem), namely

$$\tilde{\mathcal{H}}\left(\mathbf{q}^\varepsilon(t,\omega), \mathbf{p}^\varepsilon(t,\omega)\right) := \sum_{i=1}^{n}\sum_{j=1}^{k}\left\{ p_{ij}^\varepsilon(t,\omega) - \frac{\partial S}{\partial q_{ij}}\left(\mathbf{q}^\varepsilon(t,\omega)\right)\right\};$$

it obviously satisfies the hypotheses $(A_2) - (A_3)$ if the processes $\partial^2 S/\partial \mathbf{q}_i \partial \mathbf{q}_j(\mathbf{q}^\varepsilon(t,.))$ are square integrable and adapted with respect to the Brownian filtration, and if the applications $\partial^2 S/\partial \mathbf{q}_i \partial\, \mathbf{q}_j(\cdot)$ are of $C^1$ – class, locally Lipschitzian and with sublinear growth on $D_q$, for $i, j = 1, \ldots, n$.

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## 2. MAIN RESULTS

The connection between the systems (1) and (2) is the following (and this justifies why $\mathbf{q}^\varepsilon$ and $\mathbf{p}^\varepsilon$ are called stochastically perturbed coordinates and momenta). Under the hypotheses $(A_1) - (A_3)$, $\mathbf{q}^\varepsilon(t, \omega)$ converges in probability towards $\mathbf{q}(t)$ when $\varepsilon \to 0$, uniformly in $t$, i.e. for fixed $T > 0$ and every $\delta > 0$ we have

$$\lim_{\varepsilon \to 0} P^B \left\{ \omega \in \Omega, \sup_{0<t\leq T} \left\| \mathbf{q}^\varepsilon(t,\omega) - \mathbf{q}(t) \right\| > \delta \right\} = 0, \tag{3}$$

where $\|\cdot\|$ is the canonical norm on $\mathbf{R}^{kn}$ (see also Freidlin and Wentzell, 1984). The case of the momenta is similar, i. e. $\mathbf{p}^\varepsilon(t, \omega)$ converges in probability towards $\mathbf{p}(t)$ when $\varepsilon \to 0$, uniformly in $t$.

The main result concerns the problem of finding the rate of convergence in formula (3). More precisely, we have:

THEOREM 1. *Suppose that hypotheses* $(A_1) - (A_3)$ *are satisfied. Let us fix* $a$, $T > 0$, *a compact* $K_q \subset D_q$ *that contains* $\mathbf{q}_0$ *and* $\mathbf{f} : [0, T] \to \mathbf{R}^{kn}$ *an absolutely continuous function with square integrable (Lebesgue) derivative* $\mathbf{f}$. *Denote by* $\mathbf{q}^*(t)$ *the unique solution of the following differential system*

$$d\mathbf{q}^*(t) = \left[ \partial_2 \mathscr{H}(\mathbf{q}^*(t), \nabla S(\mathbf{q}^*(t))) + \partial_2 \tilde{\mathscr{H}}(\mathbf{q}^*(t), \nabla S(\mathbf{q}^*(t)))\, \mathbf{f}(t) \right] dt, \tag{4}$$

*with the initial data* $\mathbf{q}^*(0) = \mathbf{q}_0$.

*If* $P^B(.,.)$ *denotes the conditional law of the Brownian paths, then, for every* $\rho$, $r > 0$, *there exist* $\alpha_0 > 0$ *and a function* $\varepsilon_0(\alpha_0) > 0$ *such that, if the conditions*

$$\int_0^T \|\mathbf{f}(t)\|^2 dt \leq a \quad \text{and} \quad \mathbf{q}^*([0, T]) \subseteq K_q$$

*are satisfied, then*

$$P^B = \left\{ \omega \in \Omega; \sup_{0<t\leq T} \left\| \mathbf{q}^\varepsilon(t,\omega) - \mathbf{q}^*(t) \right\| \leq \rho \,\middle|\, \sup_{0<t\leq T} \left\| \varepsilon \mathbf{B}(t,\omega) - \mathbf{f}(t) \right\| \leq \alpha \right\} \geq 1 - \exp\left(-r/\varepsilon^2\right)$$

*for* $\alpha \leq \alpha_0$ *and* $\varepsilon \leq \varepsilon_0(\alpha_0)$.

*Let us fix now* $a$, $T$, $\mathbf{f}$ *as above, a compact set* $K_p \subset D_p$ *such that* $\nabla S(K_q) \subset K_p$, *and take* $b > 0$ *such that*

$$\sup_{0<t\leq T} \left\| \nabla S(\mathbf{q}(t)) \right\| \leq b.$$

*Then, for every* $\rho$, $r > 0$, *there exist* $\alpha_1 = \alpha_0(b)$ *and function* $\varepsilon_1(\alpha_1) = \varepsilon_0(\alpha_1, b)$ *such that, if the conditions*

$$\int_0^T \|\mathbf{f}(t)\|^2 dt \leq a \quad \text{and} \quad \nabla S(\mathbf{q}^*([0, T])) \subseteq K_p$$

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*are satisfied, then*

$$P^{\mathbf{B}}=\left\{\omega\in\Omega:\sup_{0<t\leqslant T}\left\|\mathbf{p}^{\varepsilon}(t,\omega)-\nabla S(\mathbf{q}^{*}(t))\right\|\leqslant\rho\left|\sup_{0<t\leqslant T}\left\|\varepsilon\mathbf{B}(t,\omega)-f(t)\right\|\leqslant\alpha\right.\right\}\geqslant 1-e^{-r/\varepsilon^{2}}$$

for $\alpha\leqslant\alpha_1$ *and* $\varepsilon\leqslant\varepsilon_1(\alpha_1)$.

The proof is based on the "large deviations" technique, namely on the following plan, see Azencott (1981), Stoica (1995):

(i) Notice that the solution $\mathbf{q}^{*}(t)$ of equation (4) is absolutely continuous and, if $\mathbf{f}$ satisfies the restriction $\int_{o}^{T}\left\|\dot{\mathbf{f}}(t)\right\|^{2}\mathrm{d}t\leqslant a$, then the correspondence $(\mathbf{q}_0,\ \mathbf{f})\to\mathbf{q}^{*}$ is continuous.

(ii) The conditional probabilities we have to estimate essentially reduce to similar probabilities fot the martingale part of the semimartingale $\mathbf{q}^{\varepsilon}(t,\omega)-\mathbf{q}^{*}(t)$, because the bounded variation part of this semimartingale lies, in norm and almost surely, in the compact set considered in the theorem.

(iii) Consider the martingale part from (ii) stopped at the first instant when the process $\mathbf{q}^{\varepsilon}(t,\omega)-\mathbf{q}^{*}(t)$ becomes greater than the given value $\rho$, and afterwards we estimate its law by means of the exponential inequality for martingales on a time interval $[0,\ s]$, on which the correspondence from (i) is well-defined and continuous.

(iv) Using the Markov property of the solution of system (2), we extend the estimations obtained in (iii) on time intervals of the form $[t,\ t+s]$.

(v) Performing a discretization of an arbitrarily time interval $[0,\ T]$, we "sum" the estimations obtained in (iv) on the corresponding subintervals.

In particular, for $\mathbf{f}=0$, we deduce from Theorem 1 the following result:

COROLLARY 2. *Under the hypotheses of Theorem 1 and for every* $\rho,\ T,\ r>0$, *there exists* $\alpha_0>0$ *and a function* $\varepsilon_0(\alpha_0)>0$ *such that*

$$P^{\mathbf{B}}\left\{\omega\in\Omega:\sup_{0<t\leqslant T}\left\|\mathbf{q}^{\varepsilon}(t,\omega)-\mathbf{p}(t)\right\|\leqslant\rho\left|\sup_{0<t\leqslant T}\left\|\mathbf{B}(t,\omega)\right\|\leqslant\frac{\alpha}{\varepsilon}\right.\right\}\geqslant 1-e^{-r/\varepsilon^{2}}$$

*for* $\alpha\leqslant\alpha_0$ *and* $\varepsilon\leqslant\varepsilon_0(\alpha_0)$. *In addition, there exist* $\alpha_1=\alpha_0(b)$ *and a function* $\varepsilon_1(\alpha_1)=\varepsilon_0(\alpha_1,\ b)$ *such that*

$$P^{\mathbf{B}}\left\{\omega\in\Omega:\sup_{0<t\leqslant T}\left\|\mathbf{p}^{\varepsilon}(t,\omega)-\mathbf{p}(t)\right\|\leqslant\rho\left|\sup_{0<t\leqslant T}\left\|\mathbf{B}(t,\omega)\right\|\leqslant\frac{\alpha}{\varepsilon}\right.\right\}\geqslant 1-e^{-r/\varepsilon^{2}}$$

*for* $\alpha\leqslant\alpha_1$ and $\varepsilon\leqslant\varepsilon_1(\alpha_1)$.

*Interpretations.* – The solution $\mathbf{q}^{*}(t)$ of system (4) appears as a deterministic perturbation of the coordinates $\mathbf{q}(t)$ on $D_{\mathbf{q}}$, and $\nabla S\ (\mathbf{q}^{*}(t))$ as a deterministic perturbation of the momentum $\mathbf{p}(t)$ on $D_{\mathbf{p}}$, hence Theorem 1 yields an asymptotic estimation (uniformly in $t\in(0,\ T]$) of the difference between deterministic and stochastic perturbations of the coordinates and momenta.

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The result obtained in the Corollary 2 strenghs the link given by formula (3) between the systems (1)–(2), because it offers an asymptotic estimation of the rate of convergence (in probability, when $\varepsilon \to 0$ and uniformly in $t \in (0, T]$) of the difference between the coordinates $\mathbf{q}(t)$, the momenta $\mathbf{p}(t)$ and their stochastic perturbations $\mathbf{q}^\varepsilon(t, \omega)$, $\mathbf{p}^\varepsilon(t, \omega)$.

More precisely, for small $\varepsilon$, $\alpha > 0$, *if we neglect an event of small probability (smaller than* $\exp(-r/\varepsilon^2)$, *with r sufficiently large) then:*

i) *in the case of coordinates, the presence of* $\varepsilon \mathbf{B}(\cdot,\omega)$ *in the uniform ball centered in* $\mathbf{f}(\cdot)$ *and of radius* $\alpha$, *implies the presence* $\mathbf{q}^\varepsilon(\cdot,\omega)$ *in the uniform ball centered in* $\mathbf{q}^\varepsilon(\cdot)$ *and of radius* $\rho$ (the case of the momenta is similar);

ii) *the presence of* $\mathbf{B}(\cdot,\omega)$ *in the centered uniform ball of radius* $\alpha/\varepsilon$, *implies the presence of* $\mathbf{q}^\varepsilon(\cdot,\omega)$ *(and of* $\mathbf{p}^\varepsilon(\cdot,\omega)$ *respectively) in the uniform ball of radius* $\rho$ *and centered in* $\mathbf{q}(\cdot)$ *(in* $\mathbf{p}(\cdot)$ *respectively).*

We can go further with the estimations obtained above and apply them to the problem of estimating the probability that the stochastically perturbed coordinates or impulse lie in some given admissible domain.

**Proposition 3.** *Suppose that the hypotheses of Theorem 1 are fulfilled. If* $\mathbf{q}^*$ *is the solution of equation (4) for given* $\mathbf{f}$ *(absolutely continuous on* $\mathbf{R}$*), let us denote*

$$\lambda(\mathbf{q}^*) := \inf\left\{\frac{1}{2}\int_0^T \left\|\dot{\mathbf{f}}(t)\right\|^2 dt,\ \mathbf{q}^*(0) = \mathbf{q}_0\right\}.$$

*Let* $L_q \subseteq D_q$ *be a domain such that* $\mathbf{q}_0 \in L_q$ *and* $\partial L_q$ *(the boundary of* $L_q$*) equals the boundary of the adherence of* $L_q$*. Then, for fixed* $t > 0$, *we have*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log P^{\mathbf{B}}\left\{\mathbf{q}^\varepsilon(t,\omega) \in L_q\right\} = -\inf\left\{\lambda(\mathbf{q}^*),\ \mathbf{q}^*(t) \in L_q \cup \partial L_q\right\}. \qquad (5)$$

*In addition, if* $\nabla S$ *is an open application, then* $L_p := \nabla S(L_p)$ *is a domain in* $D_p$ *such that* $\mathbf{p}_0 \in L_p$ *and*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log P^{\mathbf{B}}\left\{\mathbf{p}^\varepsilon(t,\omega) \in L_p\right\} = -\inf\left\{\lambda(\nabla S(\mathbf{q}^*)),\ \mathbf{q}^*(t) \in L_q \cup \partial L_q\right\}.$$

The proof is based on the following plan:

(i) Let us suppose the following technical assumption: the $(n + k, n + k)$ matrix application

$$\begin{pmatrix} \partial_2 \widetilde{\mathscr{H}} & 0 \\ 0 & (\partial_1 \widetilde{\mathscr{H}})^t \end{pmatrix}$$

is invertible on $D_q \times D_p$; then it is straightforward to check that the above explicit formula for $\lambda(\mathbf{q}^*)$ gives the correct rate functional for the stochastically perturbed coordinates associated to the system (2), see Stoica (1995). To avoid

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the above technical assumption (which is not always satisfied, see Bismut's example of perturbing Hamiltonian), we must use the arguments from the Azencott (1980).

(ii) Transporting continuously the large deviations principle for the Brownian motion process, and using the Cramer transform given above, we obtain a large deviation principle for the stochastic system (2).

(iii) The equality between the boundaries of $L_q$ and that of the adherence of $L_q$ is a sufficient condition to obtain equality in the large deviation principle obtained in (ii) (this principle being in fact a double inequality) and, passing to the limit, this implies formula (5).

(iv) Using the fact that the application $\nabla S$ is of $C^1$–class, the last formula follows from (5) by continuous transformation of the rate functional for $\mathbf{q}^\varepsilon$.

*Comment.* In order to compute the expression in the right-hand side of formula (5), one usually applies the classical variational method for the function

$$V(t, x, y) := \inf \{\lambda(\mathbf{q}^*), \mathbf{q}^*(0) = x, \ \mathbf{q}^*(t) = y\}$$

to which we associate the deterministic Hamilton-Jacobi equation, together with the conditions $V(0, 0, x) = 0$, $V(t, x, y) \geqslant 0$.

## 3. APPLICATION TO (CELESTIAL) MECHANICS

The typical examples of applications of the above estimations are the exactly harmonic systems, e. g. the isochrone systems, the free rigid body, the harmonic oscillator, the two-body problem. From both mechanical and stochastic points of view, all these examples are equivalent: they are integrable and the estimations obtained in this paper are the same for all of them. For instance, the Newtonian Hamiltonian in the relative planar two-body problem ($k = 1$, $n = 2$) takes the form

$$\mathscr{H} = \mathscr{H}(\mathbf{p}, \mathbf{q}) = \frac{1}{2}\left(p_1^2 + \frac{p_2^2}{q_1^2}\right) - \frac{\mu}{q_1},$$

(in polar coordinates), where $\mu > 0$ is the gravitational parameter, and $\mathbf{q} = (q^1, q^2)$, $\mathbf{p} = (p^1, p^2)$. One may take (in the elliptic case)

$$D_q := \left(\frac{-\mu + \sqrt{\mu^2 + 2hC^2}}{2h}, \frac{-\mu - \sqrt{\mu^2 + 2hC^2}}{2h}\right) \times \mathbb{R},$$

where $C$ is the angular momentum constant and $h < 0$ is the energy constant: a complete integral of the associated Hamilton-Jacobi equation is given by $-ht + Cq_2 + f(q_1, h, C)$ (see also condition $(A_1)$), with $f$ of $C^2$–class, computed explicitly by quadratures; for details, see Stoica and Stoica (1994).

Let us to return to Corollary 2, in which we take $r = 1$ and suppose that $\varepsilon$ is very small with respect to the size of the applications $\alpha_0$, $\alpha_1$. Inspired by the possibility of linearizing the application $S$ on a small time interval $(0, T]$ in the above exactly harmonic systems, we may suppose that the dependence of $\rho$ in the applications $\alpha_0$, $\alpha_1$ is linear. We thus obtain the following practical consequence with sharp estimates:

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*Let* $\rho$, $\varepsilon$, $T > 0$ *be given. Suppose that the stochastic perturbation (represented in our model by the Brownian motion process) has the order of* $\rho/\varepsilon$; *by Ikeda and Watanabe (1989), this happens with probability at least equal to* $1 - 2\exp\times(-\rho^2/2\varepsilon^2 T)$, *i. e.*

$$P^{\mathbf{B}}\left\{\sup_{0<t\leq T}\left\|\mathbf{B}(t,\omega)\right\| \leq \frac{\rho}{\varepsilon}\right\} \geq 1 - 2\exp\left(-\rho^2/2\varepsilon^2 T\right). \tag{6}$$

*Then, by Corollary 2, the difference between coordinates, momenta and their stochastic perturbations is smaller than* $\rho$, *with the probability at least equal to* $1 - \exp(-1/\varepsilon^2)$ *i. e.*

$$P^{\mathbf{B}}\left\{\sup_{0<t\leq T}\left\|\mathbf{q}^{\varepsilon}(t,\omega) - \mathbf{q}(t)\right\| \leq \rho,\ \sup_{0<t\leq T}\left\|\mathbf{p}^{\varepsilon}(t,\omega) - \mathbf{p}(t)\right\| \leq \rho\right\} \geq 1 - \exp\left(-1/\varepsilon^2\right). \tag{7}$$

*In particular, for the efficiency of the estimates given in formula* (7), *it follows from formula* (6) *that we must choose "small" values for* $\varepsilon$ *and* $T$, *and "not too large" values for* $\rho$.

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# PERIOD CHANGES OF SATTELITE CIRCULAR MOTION DUE TO ODD ZONAL HARMONICS OF GEOPOTENTIAL

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*Abstract.* The initially circular motion of an artificial satellite in the terrestrial gravitational field is being considered. The first order difference between the nodal period and the Keplerian one caused by all odd zonal harmonics of the geopotential is analytically determined. The effect of every such harmonic can be pointed out separately.

*Key words*: artificial satellites – terrestrial gravitational field.

## 1. INTRODUCTION

The perturbed orbital motion of artificial satellites in the terrestrial gravitational field constitutes a well studied problem, which however keeps its present interest. Very significant contributions to this problem have been brought by King-Hele (1958), Brower (1959), Kozai (1959), Iszák (1960, 1961), Zhonglovich (1960a, b), Kaula (1966), and many others. Such studies take into account only one or few harmonics of the geopotential.

Pal et al. (1994) consider the influence of all zonal harmonics of the geopotential on the orbital elements of a satellite moving on an initially circular trajectory. In this paper we shall determine the first order effects caused by the odd zonal harmonics in the nodal period of a satellite in an initially circular orbit. Given the particular kind of perturbations induced by the odd zonal harmonics in the orbital parameters (see e. g. Mioc 1987), their influence deserves to be studied separately.

## 2. BASIC METHODS AND EQUATIONS

Since we deal with the nodal period, it is natural to choose as independent variable the argument of latitude $u$. Therefore we start from the Newton-Euler equations written with respect to this variable, in the form (e. g. Mioc 1991)

$$\begin{aligned}
dp/du &= 2\,(Z/\mu)\,r^3\,T,\\
dq/du &= (Z/\mu)\,\{r^3 kBCW/(pD) + r^2 T\,[r(q+A)/p + A] + r^2 BS\},\\
dk/du &= (Z/\mu)\,\{-r^3 qBCW/(pD) + r^2 T\,[r(k+B)/p + B] - r^2 AS\},\\
d\Omega/du &= (Z/\mu) r^3\, BW/(pD)\\
di/du &= (Z/\mu)\,r^3 AW/p,\\
dt/du &= Zr^2/\sqrt{\mu p},
\end{aligned} \tag{1}$$

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where $Z = [1 - r^2 C(d\Omega/dt) / \sqrt{\mu p}\,]^{-1}$, $\mu$ = Earth's gravitational parameter, $r$ = geocentric radius vector, $p$ = semilatus rectum, $A = \cos u$, $B = \sin u$, $C = \cos i$, $D = \sin i$ ($i$ = inclination), $\Omega$ = longitude of the ascending node, $q = e \cos \omega$, $k = e \sin \omega$ ($e$ = eccentricity, $\omega$ = argument of perigee), $S$, $T$, $W$ = radial, transverse, and binormal components of the perturbating acceleration, respectively.

The change of $y \in \{p, q, k, \Omega, i\}$ in the interval $[u_0, u]$ is

$$\Delta y = \int_{u_0}^{u} (dy / du)du, \tag{2}$$

with the integrands given by (1). The integrals are estimated by successive approximations, with $Z \approx 1$. We limit the process to the first order approximation.

The first order difference $\Delta T_\Omega$ between the (perturbed) nodal period and the corresponding Keplerian period (for $u_0 = u$) is found, by using Zhongolovich's (1960a) method, to be the sum $\Delta T_\Omega = \sum_{s=1}^{4} I_s$, where (Mioc 1980a)

$$I_1 = \frac{3}{2} p_o^{1/2} \mu^{-1/2} \int_0^{2\pi} (1 + q_0 A + k_0 B)^{-2} \Delta p\, du,$$

$$I_2 = -2 p_o^{3/2} \mu^{-1/2} \int_0^{2\pi} (1 + q_0 A + k_0 B)^{-3} A \Delta q\, du, \tag{3}$$

$$I_3 = -2 p_o^{3/2} \mu^{-1/2} \int_0^{2\pi} (1 + q_0 A + k_0 B)^{-3} B \Delta k\, du,$$

$$I_4 = \int_0^{2\pi} \{\partial [r^4 C(d\Omega / dt) / (\mu p)] / \partial \sigma\} \sigma\, du.$$

Here $\sigma$ is a small parameter featuring the perturbing factor, the subscript "0" refers hereinafter to $u = u_0$, while $\Delta y$ are the first order changes determined with (2).

## 3. PERTURBING ACCELERATION

Consider that the Earth has geometric and dynamic symmetry with respect to its rotation axis, hence only the zonal harmonics of the geopotential will be taken into account. Limiting ourselves to the odd zonal harmonics, the perturbing function has the expression (Zhongolovich 1960b)

$$U = \mu \sum_{t=1}^{\infty} c_{2t+1} (R^{2t+1} / r^{2t+2}) P_{2t+1}(\sin \varphi), \tag{4}$$

where $R$ = mean terrestrial equatorial radius, $c_{2t+1}$ = dimensionless coefficients of odd zonal harmonics, $\varphi$ = latitude, $P_{2t+1}$ = Legendre polynomials. The components of the perturbing acceleration are

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$$S = \partial U/\partial r,$$
$$T = (1/r)(\partial U/\partial \varphi)(A/B)\tan\varphi, \tag{5}$$
$$W = (1/r)(\partial U/\partial \varphi)C\sec\varphi.$$

Performing the calculations, taking into account the fact that $\sin\varphi = DB$, and abridging

$$F_{2t+1,m} = \frac{(-1)^{m+1}(2t+2)(4t-2m+2)!}{2^{2t+1}m!(2t-m+1)!(2t-2m+1)!}, \tag{6}$$

$$G_{2t+1,m} = \frac{(-1)^{m}(2t-2m+1)(4t-2m+2)!}{2^{2t+1}m!(2t-m+1)!(2t-2m+1)!},$$

expressions (5) become

$$S = \mu\sum_{t=1}^{\infty} c_{2t+1}(R^{2t+1}/r^{2t+2})\sum_{m=0}^{t} F_{2t+1,m}D^{2t-2m+1}B^{2t-2m+1},$$

$$T = \mu\sum_{t=1}^{\infty} c_{2t+1}(R^{2t+1}/r^{2t+2})\sum_{m=0}^{t} G_{2t+1,m}D^{2t-2m+1}AB^{2t-2m}, \tag{7}$$

$$W = \mu\sum_{t=1}^{\infty} c_{2t+1}(R^{2t+1}/r^{2t+2})\sum_{m=0}^{t} G_{2t+1,m}CD^{2t-2m}B^{2t-2m}.$$

## 4. NODAL PERIOD VARIATION

Replacing (7) in the first equation (1) considered separately, and taking into account the fact that the initial orbit is circular ($r_0 = p_0$, $q_0 = k_0 = 0$), we get the integrands of (2) expressed only in terms of explicit functions of $u$ (though $A$ and $B$) and quantities considered constant over one revolution of the satellite. Performing the integrals (2) for $p$, $q$, $k$, we obtain

$$\Delta p = P - P_0, \quad \Delta q = Q - Q_0, \quad \Delta k = K - K_0, \tag{8}$$

where

$$P(u) = 2p_0\sum_{t=1}^{\infty}\sum_{m=0}^{t}\alpha_{tm}f_{tm}^{(1)}B^{2t-2m+1},$$

$$Q(u) = \sum_{t=1}^{\infty}\sum_{m=0}^{t}\alpha_{tm}\left[f_{tm}^{(2)}A\left(B^{2t-2m-1} + \sum_{j=1}^{t-m-1}K_{tmj}^{(2)}B^{2t-2m-2j-1}\right) + \right.$$
$$\left. + f_{tm}^{(3)}A\left(B^{2t-2m+1} + \sum_{j=1}^{t-m}K_{tmj}^{(2)}B^{2t-2m-2j+1}\right) + f_{tm}^{(4)}u\right], \tag{9}$$

$$K(u) = \sum_{t=1}^{\infty}\sum_{m=0}^{t}\alpha_{tm}f_{tm}^{(3)}B^{2t-2m+2},$$

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and $X_0 = X(u_0)$, $X \in \{P, Q, K\}$. In (9) we abridged

$$\alpha_{tm} = c_{2t+1}(R/p_0)^{2t+1} D_0^{2t-2m+1},$$

$$f_{tm}^{(1)} = G_{2t+1,m}/(2t-2m+1),$$

$$f_{tm}^{(2)} = -G_{2t+1,m}/(t-m),$$

$$f_{tm}^{(3)} = (2G_{2t+1,m} - F_{2t+1,m})/(2t-2m+2),$$

$$f_{tm}^{(4)} = 2G_{2t+1,m}(2t-2m-1)!!/[2^{t-m}(t-m)!] + \tag{10}$$

$$+(F_{2t+1,m} - 2G_{2t+1,m})(2t-2m+1)!!/[2^{t-m+1}(t-m+1)!],$$

$$K_{tmj}^{(1)} = \frac{(2t-2m-1)(2t-2m-3)...(2t-2m-2j+1)}{2^j(t-m-1)(t-m-3)...(t-m-j)_-},$$

$$K_{tmj}^{(2)} = \frac{(2t-2m+1)(2t-2m-1)...(2t-2m-2j+3)}{2^j(t-m)(t-m-1)...(t-m-j+1)}.$$

Now, introducing (8)–(9) in the first three expressions (3) and performing the integrals, then calculating $I_4$ by means of the fourth and last equations (1) and the third expression (7), the part of the small parameter $\sigma$ being played by $c_{2t+1}$, and finally performing the sum $I_1 + I_2 + I_3 + I_4$, we get

$$\Delta T_\Omega = -6\pi p_0^{3/2}\mu^{-1/2}\sum_{t=1}^{\infty}\sum_{m=0}^{t}\alpha_{tm} f_{tm}^{(1)} B^{2t-2m+1}. \tag{11}$$

This is the first order difference between the nodal period and the corresponding Keplerian one caused by all odd zonal harmonics of the geopotential for initially circular orbits.

The effect of every odd zonal harmonic can be pointed out by particularizing the index $t$. For instance, the third zonal harmonic ($t = 1$) causes a difference

$$\Delta T_\Omega = 3\pi c_3 R^3 p_0^{-3/2}\mu^{-1/2} D_0 B_0(-5D_o^2 B_o^2 + 3). \tag{12}$$

where we took into account notations (10). In this way we recover the expression found by Mioc (1980b).

Although our results are restrictive enough (first order effects, initially circular orbits), they have a great avantage, allowing the determination of the influence of any zonal harmonic (or group of such harmonics). Further cases (even harmonics, noncircular orbits, etc.) will be tackled elsewhere.

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# MEASUREMENT AND ASTROMETRIC REDUCTION OF CCD IMAGES IN BUCHAREST

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*Abstract:* The paper presents the methods of measurement and astrometric reduction of the CCD images made at Bucharest Astronomical Observatory beginning with 1995. We present the programs for the automatic measurement and the astrometric reduction methods, as well as the accuracy of the astrometric positions obtained.

*Key words*: CCD, Astrometry.

## 1. ASTROMETRIC CCD OBSERVATIONS PERFORMED IN BUCHAREST

Photographic astrometry has old traditions at Bucharest Astronomical Observatory. Thus, beginning with 1930, exposures of photographic plates (24 × 24 cm, namely 2.3 × 2.3 degrees) were made with the Prin-Merz double refractor ($f = 6$ m, $\Phi = 38$ cm), for astrometric purposes. The database which identifies the photographic observations made in the period 1930–1993 at the observatory in Bucharest (Vass et al., 1994; Vass, 1994) contains 10 571 photographic plates made in that period, from among which 8 234 have asteroids.

In Romania the first CCD astronomical camera was mounted at the Astronomical Institute of the Romanian Academy in Bucharest in the autumn of 1994, within the international collaboration occasioned by the PHESAT'95 campaign, whose main goal was to observe the mutual phenomena in the system of Saturn's satellites. The observations within this campaign began in the summer of 1995.

The camera was mounted on the visual telescope of the Prin-Merz astrograph, with a 38 cm diameter and a 6 m focal distance.

The time basis is provided by the time service of Bucharest Observatory, by means of a chronographic card designed by M. Dequin and F. Chollet (of Paris Observatory) and made in Bucharest. To make the best of the use of the chronographic card, we resorted to a specially designed program (TSR – Terminate, Stay Resident), meant to ensure the periodical synchronization of the real time clock of the computer (with a one second or one minute step).

In view of the fact there were only approximately 30 nights when observations were actually carried out for PHESAT'95 and also the fact that each night dozens or hundreds of auxiliary images (images for flat-field and thermal noise corrections, for various windows used) were made, it results that the number of auxiliary images made during this time amounts to no less than 2 000–3 000. To these, the images themselves, over 4 000 in number, have to be added. From among the latter ones, 2 670 are astrometric images, containing



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images of Saturn's satellites (2 520) and of the M15 cluster (150), which are used to orientate the camera.

As the observation data accumulated, the necessary programs of preliminary processing were elaborated (we refer here to the pre-processing and to the measurement programs). The astrometric reduction programs began to be elaborated only towards the end of the campaign and were finished in the summer of 1996.

In the same period, as a result contacting the "European Asteroidal Occultation Network" (EAON) in 1994, one of the authors of the present paper (O. Văduvescu) made the first attempts to observe the occultation of the stars by the asteroids. The actual occultations being rare, we have come to the conclusions that the respective observations should be completed with astrometric CCD observations of the asteroids, in the moments when those are close to the astrometric catalogues stars.

A first attempt to observe the occultation of a star was made on 29 August, when (7) IRIS occultated (but not in Bucharest !) the PPM 92838 star. So far, three approaches of the stars in the PPM catalogue by asteroids have been observed.

## 2. CCD IMAGES MEASUREMENT

The astronomical CCD camera Hi-SIS 22, with which the images were obtained, has a detector of KODAK KAF–0400 type. Under the conditions in Bucharest (unsatisfactory transparency and appreciable light pollution), the camera has been used in "binning 2" mode, which had as a result images of 384 × 256 pixels. In this way, as sensibility grows, the integration time for each image can be reduced. We note a primary pixel with a $p$ and an actual pixel with a $P$ (in binning 2); in the astrograph + CCD system we have the correspondances given in Table 1.

The acquisition and processing programs accompanying the Hi–SIS camera are QMiPS (Quick MiPS) and WINMiPS (Buil, 1995; Buil et al.,). At the same

*Table1*

| | " | mm | μ | p | P |
|---|---|---|---|---|---|
| 1" = | 1 | 0.029089 | 29.089 | 3.23 | 1.62 |
| 1mm = | 34.377 | 1 | 1000 | 111.11 | 55.562 |
| 1μ = | 0.034377 | 0.001 | 1 | 0.11111 | 0.0555 |
| 1p = | 0.3093 | 9 | 0.009 | 1 | 0.5 |
| 1P = | 0.618786 | 18 | 0.018 | 2 | 1 |

time with the camera, we received from the Bureau des Longitudes in Paris also a software made by F. Colas, named ASTROL (Colas et al.,1992).

QMiPS and ASTROL offer the user many useful functions in the acquisition, pre-processing and measurement of the CCD images; they also include an "interpretor" which allows the user to write specific programs himself, using the primitive functions and a number of instructions and control structures.

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The most efficient measurement function of the stellar CCD images, of ASTROL, is CENTRO; it analyses the situation in a square area of a given size, centered on the position of the current cursor and performs the following operations:

- evaluates the intensity of the background, which it represents by means of a two variables function, of 2nd degree maximum;
- eliminates the background from the evaluation of the (central) stellar body flux;
- shows the distribution of the stellar body intensity through a Gaussian function of two variables;
- integrates the flux coming from the star, throughout the entire domain considered;
- supplies the following information, placing it in ten global variables: the abscissa of the star (namely, of the determined gaussian's symmetry axis); the ordinate of the star; the flux; the error of the abscissa; the error of the ordinate; the flux error (%); the full width at half maximum (the seeing) on Ox; the full width at half maximum (the seeing) on Oy; the O–C in Gaussian's determination; the signal/noise ratio.

To perform these operations, CENTRO needs the following input parameters:
- the side of the analysed area (17, 15, 13, 11, 9, 7 and 5 pixels);
- the degree of the polynomial representing the background (0,1,2).

A very useful, auxiliary function of measurement is COUPE, which displays two profiles of the image, corresponding to the parallels to each axis, driven through the current cursor; we have used this function to fix the current cursor as close as possible to the center of the bodies of stellar aspect, before activating the CENTRO function.

## 3. AUTOMATIC MEASUREMENT PROGRAMS

As a rule, bodies of stellar type image can be measured on any image by means of the primitive procedure CENTRO (we resorted to "the preliminary centering by means of the COUPE function"). However, the manual performance of the preliminary manoeuvres of centering each stellar image can become tiresome and inconvenient because of the great amount of time needed, especially if on an image there are many such objects to be measured and also if there are series with numerous images.

In such cases, it is compulsory to write programs which identify automatically the objects concerned. Within the PHESAT'95 campaign several such programs were made.

After the measurements were carried out, the images were transferred on a streamer cassette, keeping on the hard disk of the computer only the files with the results of the measurements; in this way the blockage of the computer due to the consuming of the entire space of the hard disk, which is actually sufficient only for a few nights of observations, could be avoided.

Obviously, the measurement itself has to be made on a pre-processed image. Consequently, all our programs include a sequence intended for the pre-processing

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of the images to be measured; also, an important part of the programs is planned for the best visualization of the objects measured, visualization which adjusts itself interactively.

All programs of automatic measurement include also the possibility of appropriate adjustment of the CENTRO function parameters.

### 3.1. Measurement of Planetary Satellites Positions

The measurement programs for the positions of Saturn's satellites were elaborated starting from the idea that, in a series of images, between two successive images there are no essential movements of satellites, so that the CENTRO function can identify the present position of the satellite on the basis of the previous position of the cursor.

Consequently, on the first image of the series all satellites that can be visualized are spotted, the program memorizes their names and positions, and on the following image the cursor is placed automatically, by the program, on the respective positions; after that, CENTRO is started.

If one of the satellites undergoes a displacement which prevents the CENTRO function from making a correct determination, the program senses this and asks the user to make the identification manually. The program uses the relative positions of the satellites as against the first measured satellite; this manner makes it possible, if the position of the instrument has been modified between two images of the series, to come back to normal (automatic determinations) after the manual determination of the first satellite with an absolute position modified in the image.

After the measurement of the first image of the series, for which an adequate visualization of each satellite is necessary, the program can perform measurements without the satellites being actually visible on the screen; the actual visualization is necessary only for the manual centering of the satellite.

### 3.2. Measurement of Stars in Globular Clusters

The globular clusters are objects used to determine the orientation of the CCD camera in a night of observations; they are recommended for that because they contain many stars in a small area, of some square arc minutes.

The measurement of the stars of a globular cluster raises problems of a different nature than those posed by the series of satellites images; here one has to do with a single image which contains dozens or even hundreds of stars. To manually center these stars is practically impossible, and especially the correct identification of the stars in the cluster, at each image measurement.

In the PHESAT'95 campaign the cluster M15 was used; the problems connected to the measurement were additionally complicated by the fact that the identification of the stars in the cluster by means of a reference catalogue could be made only at the end of the campaign.

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Under these conditions, in order not to leave any image not measured and thus to complicate even more the already very complicated management of the observation data, we drew up an *ad hoc* catalogue, on the basis of four reference stars which the user has to recognize and to center manually, in each image. The four reference stars, having the catalogue numbers 1–4, are chosen so that they can be found in any image of the cluster, made for the determination of the camera orientation, whatever this orientation might be. They are, therefore, included in a circular area with a diameter smaller than two arc minutes, situated at the centre of the cluser.

The program asks the user to determine manually the four reference stars; with these measurements and the corresponding data from the catalogue-file, the program calculates the coefficients of a translation and of a rotation from the *ad hoc* catalogue to the image given. Then, scanning sequentially the catalogue-file, the coordinates in the image for each star in the file-catalogue are determined; the program fixes the cursor on the respective position and activates the CENTRO function, which supplies the precise measurement data, writing them in the output file.

The stars which could not be measured are recognized by the null value of the coordinates and can be rejected after the measurement or even through the program. Usually, more than 75% of stars are actually measured; the time needed for the measurement of an image with M15 is of under 5 minutes, all manual manoeuvres included.

## 4. DETERMINATION OF CCD IMAGES ORIENTATION

CCD astrometry has several characteristics which distinguish it from the photographic one; the first one is the smallness of the field which, in our case, is of $4 \times 2.6$ arc minutes.

In such a small field, the probability of catalogue stars existing in the same field with a given object is inherently very small. Even the numer of ordinary stars is often reduced by the small integration time, which is established depending on the brightness of the observed object. Consequently, generally speaking, the determination of the absolute astronomical coordinates is out of the question; there is only the possibility of determining relative positions referred (in PHESAT '95 campaign) to one of the satellites, whose orbit is better known.

In the above mentioned context, the problem of determining the orientation of the CCD camera becomes fundamental, because the knowledge of this orientation allows the transformation of the relative position to the astronomical reference system, instead of leaving these positions in the arbitrary system of the instrument.

### 4.1. Determination Method for Camera Orientation

The transformation from the equatorial coordinates ($\alpha$, $\delta$) to the ideal (standard) ones ($\xi$, $\eta$) and the reverse are given by the gnomonic projection relationships (Smart, 1948).

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Let $(X, Y)$ be the reference system of the CCD, in which the image processing software measures the coordinates, expressed in pixels. The relation between $(\xi, \eta)$ and $(X, Y)$ is given by the mathematical model adopted for the phenomenon of image formation; the simplest model is that of the linear transformation, in which case we have:

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix} \quad (1)$$

The determination of the orientation means the determination of the coefficients in the transformation (1) between the standard and the measured coordinates, leaving aside a translation.

The method used for this purpose consists of observing a zone which does not contain the objects studied, but which contains in exchange several catalogue stars (at least two); usually, the globular clusters, which have sufficiently great star densities, are preferred.

When there are only two catalogue stars, the simplified model of rotation and translation ($e = a$, $d = -b$) can be applied.

In the case of a globular cluster there are several dozens stars which can be measured, consequently we can write the relations (1) for each star and determine the unknown coefficients, through the least squares method; an even more complicated model than that of the relations (1) can be taken into consideration.

### 4.2. Identification of Stars Measured in the M15 Cluster

To identify the stars measured in the M15 cluster we resort to "link" the cluster stars to a PPM star, through a chain (mosaic) of images containing at one end the PPM star and at the other end the four stars commonly used in the measurement of the cluster images.

First, we looked for two PPM stars to enter the same CCD image: PPM 140264 and PPM 140261; with these stars we determined the camera orientation.

The camera was positioned on the PPM star nearest to M15 (PPM 140217); we determined, after some attempts, the route of a chain of CCD images, which should link the PPM star zone to that of the cluster. We registered then successive images, so that each image should have also stars of the formerr image.

As the orientation was known, the PPM 140217 star allowed us to determine the coefficients of the translation in the relation (1) and then the equatorial coordinates of some stars which appeared also in the following image.

After four stages, we started to determine the equatorial coordinates of the four stars framing cluster M15. The respective coordinates were sufficiently accurate to allow us to identify the stars in the most complete catalogue at our disposal (Le Campion et al., 1992).

The use of the catalogue coordinates for the four identified stars allowed a new determination, this time for all the stars in the cluster measured by us; this time, the determination accuracy was of the order of a tenth of an arc second, which allowed us to identify 34 stars.

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## 5. DETERMINATIONS OF POSITIONS IN THE CCD IMAGES

### 5.1. Relative Positions

If in the CCD field there is no reference star, but there are several objects of stellar aspect, one of these objects may be chosen as a reference object and the other objects' coordinates as against this one's can be determined. If, additionally, the camera orientation was determined independently, as we showed in paragraph 4, it is possible to determine the relative astronomical coordinates of the objects measured. For this purpose, the coefficients in relation (1) were used, less those which signified a translation.

In the PHESAT'95 campaign relative positions were determined through this procedure, taking as reference object the satellite 6 (TITAN) whenever possible.

Naturally, it is better to have the cluster image, on whose basis the orientation is determined, taken from a height close to the zone where determinations of relative positions are made, to eliminate the influence of the differential refraction. If this thing cannot be accomplished, corrections can be made later on the determined positions. Given the smallness of a CCD field, the influence of differential refraction can be generally eliminated under good conditions.

### 5.2. Absolute Positions

The existence of a catalogue star in the CCD field allows the determination of the translation parameters for the respective image and, finally, the determination of the absolute astronomical coordinates of the objects measured.

Naturally, the possible position errors of the respective star will be automatically transferred also (without any possible control) to the absolute positions determined. If there are also other stars, then the problem of evaluating the astrometric reduction accuracy can also arise.

We applied this procedure for the determination of the positions of the asteroids at a sufficient closeness to a PPM star. In rarer cases, at such a closeness, in the fields there happened to be a second PPM star; this kind of situation allowed us to estimate the accuracy of the positions obtained through the various applicable procedures (the determination of orientation based on the image of a globular cluster, based on the two stars in the field, or based on the averaging of the orientations of the acquired images series).

## 6. ESTIMATION OF POSITIONS ACCURACY

The accuracy of the camera orientation can be estimated by the mean square error of stellar images reduction in a globular cluster, which is an error calculated for independent stars, not included in the determination of the coefficients of (1). In the case of the astrograph in Bucharest, the average of mean square errors at the reduction of cluster M15 ranges between 0.131 and 0.098 arc seconds in positions, which corresponds to a mean square error, for each coordinate, ranging between 0.09 and 0.07 arc seconds.

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*Table 2*

| Dates | Stars / 3σ | | Stars / 2σ | | Stars / 1.7σ | |
|---|---|---|---|---|---|---|
| 06.26.1995 | 23 | .203 | 17 | .087 | 9 | .046 |
| 06.27.1995 | 27 | .084 | 21 | .102 | 16 | .042 |
| 06.28.1995 | 29 | .159 | 24 | .114 | 16 | .076 |
| 07.03.1995 | 25 | .252 | 22 | .227 | 14 | .111 |
| 07.20.1995 | 21 | .232 | 13 | .107 | 11 | .080 |
| 07.24.1995 | 28 | .203 | 15 | .105 | 14 | .113 |
| 08.06.1995 | 21 | .190 | 15 | .056 | 13 | .044 |
| 08.23.1995 | 25 | .221 | 21 | .188 | 19 | .185 |
| 08.24.1995 | 21 | .085 | 18 | .101 | 17 | .117 |
| 09.01.1995 | 27 | .176 | 27 | .176 | 22 | .153 |
| 09.03.1995 | 18 | .190 | 17 | .145 | 15 | .126 |
| 09.13.1995 | 28 | .060 | 23 | .088 | 22 | .079 |
| Mean values | 24.4 | .175 | 19.4 | .125 | 15.6 | .098 |

Table 2 contains the mean square errors (in position) obtained at the reduction of a number of 12 images of M15, depending on the various criteria used for rejecting the stars.

The coefficients of relation (1), determined on the basis of the M15 cluster, were consequently a good basis for positions determinations, especially for the relative ones; obviously, the absolute positions are affected in an essential way by the errors of the catalogue stars used to define the origin.

The current catalogue errors (generally, quite small) have a reduced influence on the camera orientation; to this effect, we mention a fact which might seem surprising: generally, the absolute positions of some asteroids, determined on the basis of the orientation resulting from the two catalogue stars, differ very little from those determined on the basis of the orientation resulting from the M15 cluster. Naturally, in both cases the same star was used to define the origin.

These differences did not exceed $0.002^s$ in right ascension, namely 0".02 in declination, for the 5 asteroids studied. Obviously, these values are not estimates of the position errors. Table 3 presents by comparison several such positions.

The danger of using a couple of stars for orientation lies in the absence of any control possibility; any catalogue error can introduce systematic errors in the final positions determined. We came across such a situation with the couple PPM 526658–PPM 526654, which gave an orientation that was sensibly different from that given by M15 (systematic differences of 0".5); of course, we can suppose that there is a catalogue error in this couple.

## 7. CONCLUSIONS

1. The absolute astrometric positions determined in globular clusters in Bucharest have a mean square error of less than 0.09 arc seconds on each coordinate;

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2. These values are accuracy indicators for relative positions and for the absolute ones determined by means of the "exterior" orientation of the CCD images, obtained through the observation of a globular cluster;
3. The absolute astrometric positions determined on the basis of the

*Table 3*

| α (M15) | 2 stars | δ (M15) | 2 stars |
|---|---|---|---|
| $3^h45^m42^s.877$ | $42^s.878$ | $25^h35^m35^s.51$ | $35^s.52$ |
| 3 45 43.153 | 43.154 | 25 35 36.71 | 36.72 |
| 3 45 43.249 | 43.251 | 25 35 37.05 | 37.06 |
| 3 45 43.400 | 43.402 | 25 35 37.42 | 37.43 |
| 3 45 44.010 | 44.011 | 25 35 39.65 | 39.66 |
| 3 45 44.172 | 44.173 | 25 35 40.00 | 40.00 |
| 3 45 44.453 | 44.453 | 25 35 41.10 | 41.10 |
| 3 45 45.718 | 45.717 | 25 35 45.38 | 45.38 |
| 3 45 45.757 | 45.756 | 25 35 45.40 | 45.40 |
| 3 45 45.794 | 45.793 | 25 35 45.34 | 45.34 |
| 3 45 45.883 | 45.883 | 25 35 45.74 | 45.73 |
| 3 45 45.920 | 45.919 | 25 35 46.00 | 46.00 |
| 3 45 46.005 | 46.004 | 25 35 46.25 | 46.25 |
| 3 45 46.046 | 46.045 | 25 35 46.44 | 46.44 |
| 3 45 46.164 | 46.163 | 25 35 46.75 | 46.75 |
| 3 45 46.163 | 46.163 | 25 35 47.01 | 47.00 |
| 3 45 46.294 | 46.293 | 25 35 47.21 | 47.20 |

exterior orientation of the image are essentially affected by the position errors of the catalogue star used to define the origin;
4. The images orientation can be generally determined with a satisfactory accuracy, by means of quite a small number of stars (even two!), but without any possibility of the accuracy control.

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# TIME RECORDING IN CCD OBSERVATIONS

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*Abstract.* This paper presents some methods improving the precision of time recording and time measurements in the image acquisition technique by CCD, using the PCs equipped with chronographic cards. At the Astronomical Institute of the Romanian Academy a chronographic card designed by M. Dequin and F. Chollet, Paris Observatory, was used. The results show that the accuracy can become of the order of the commutation time of the CCD camera's shutter.

*Key words*: CCD, image acquisition, programming.

The utilization of the CCD cameras in astronomical observations permits to obtain good accuracies for astrometric positions (of about 0.1 arcsec) or phenomena, but because usually CCD cameras are used with computers like the PC the time recording precision of the image obtained can be easily affected by errors.

There are two sources of primary errors, both with technical origin. The first is given by the delay between the "opening" of the CCD's shutter (start of integration) and the moment when the time is recorded for the current exposure in the acquisition software. The second error is given by the PC's operating system real time clock, which is usually different from the true time, and can fluctuate for seconds in intervals of a few hours.

We described below these errors and the methods of elimination.

The second type of error given by the operating system real time clock can be easily removed using a time base signal and a chronographic card. Because the price for a GPS card is too high to endow each PC with such a card, it is easier to use a unique GPS card or a time base standard which can generate a time base signal, and many cards for the PCs. At Bucharest Observatory, we used two signals generated by the time base standards Rhode & Schwartz, one of 10KHz, and the other of 1Hz, synchronized with the UT1 second. Inside the PC we installed a chronographic card and a specific software, designed at Paris Observatory by M. Dequin and F. Chollet. Initially, this card was designed for measurements of time and time recording with an astrolabe. It contains a counter of 32 bits (4 × 8 bits), which counts the signal of 10KHz, and two inputs for external events, both of which can generate hardware interrupts which activate the procedure of time recording in this software. One of these inputs is driven by the UT1 second (1Hz) which enures the synchronization of the 32 bits counter, and at the software start, performs just once the setting of the PC's real time clock.

We used this card to set the real time clock of the PC connected to the CCD camera, the software for image acquisition getting the PC time for each image. We found that in a range of a few hours the RTC of the OS makes many



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transitions of about 1–3 seconds, which is equal with the order of the time exposure, and can compromise an entire night of observations. It is easy to understand the reasons for why the RTC jumps, if we take into consideration the PC hardware and software equipments: only a CMOS circuit which keeps the date, time and other technical parameters and a special counter for time with a frequency of 32768 Hz, and when the PC is power on, another time counter with a frequency of 1.82 MHz (Intel Corp., 1993), all of these driven by the OS services, usually called DOS functions (for Microsoft DOS) (Duncan, 1986, Hogan,1988, Microsoft Corp., 1993) which perform the real time managements. All these equipments are not sufficient for precise time managements, for example the DOS functions can ensure only a 55ms precision (Rollins, 1989), as there are many other undocumented hardware and software bugs (Rollins, 1989).

We chose to set the PC's RTC with UT1 by using two inputs of the chronographic card, driven by the time base 10KHz signal and the 1Hz (UT1 second), through a "Terminate, Stay Resident" (TSR) software, which we especially developed, activated by hardware interrupts (Rollins,1989, Intel Corp., 1993) over a period of 1 second (UT1), or 60 seconds. The software gets the time registered by the 32 bits counter and calculates the current second, then sets the PC's RTC using this new value. From the chronographic card the TSR reads a value such as:

$$C = (count1 + count2 \times 256 + count3 \times 65536 + count4 \times 16777216)/10000$$

which corresponds to a time elapsed from an origin. If the origin is chosen as the second 0 of one minute, then the current second is:

$$Sec = C \bmod 60$$ sec. and fractions of second.

The DOS function SET_DOS_TIME need like arguments the hour, minute, second, and hundredths of a second. It is easy to compute the last two arguments:

$$S = Trunc(Sec)$$

$$S100 = Trunc(Sec \times 100) - S \times 100$$

The TSR software uses many DOS functions, and whenever such a software is programmed many precautions are needed, because an unadequate utilization can lead to the instability of the system (Borland International, 1993).

We made some time measurements to check how the system works before and after the installation of the TSR. For this we used the CCD camera to take images of short integration time (0.01 second – the shutter had a mechanical inertia and this was the lowest value possible), of a digital watch driven by the time base signal UT1 (1 Hz), and we calculated the differences between the time recorded for each image and the "photographed" time. Because the watch didn't display fractions of a second, we developed a model to evaluate these results, and we used the MATH-CAD software to process them. The "stair" aspect (figures 1, 3) is given by the delay between two successive images and the fraction of a second lost at every image (because the watch didn't display the fraction of a second). These fractions gather up, and when they exceed greater than 1 a leap down appears. The minimum of the curve gives the correction for the time recorded by the CCD software for each image. If the camera and acquisition software don't introduce any delay between the opening of the shutter and the time recording, the minimum of the curve is placed to 0 everywhere except for some points where errors like the transition of digits occur on the watch display or mechanical inertia.

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Before the installation of TSR, in one hour a transition of about 1 second for the PC's RTC was observed (Fig. 1), but after the installation of the TSR, all minimums were of the same value, namely shifted from zero with a constant value depending on the camera and the acquisition software (Fig. 3). This value is a correction which must be made at the time recorded for an image and represents the first type of error, as we already mentioned at the beginning of this paper. Removing this type of error implies the precise determination of the correction, and, after that, modifications in the sources of the acquisition software, because it is often impossible to modify the CCD camera hardware.

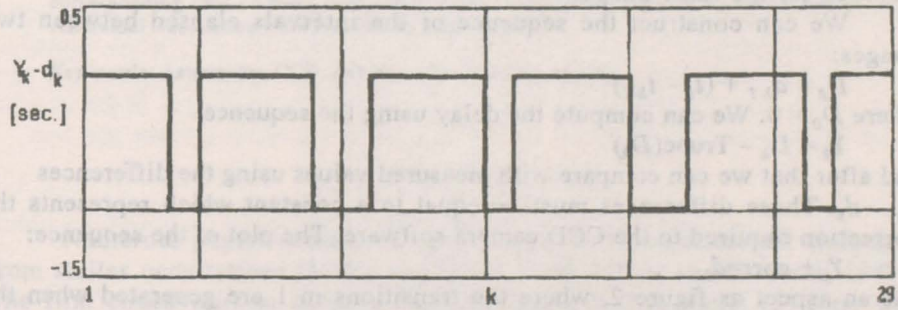



Fig. 1 – The delay between the PC's RTC and the true time. 100 images were taken in 840 seconds.

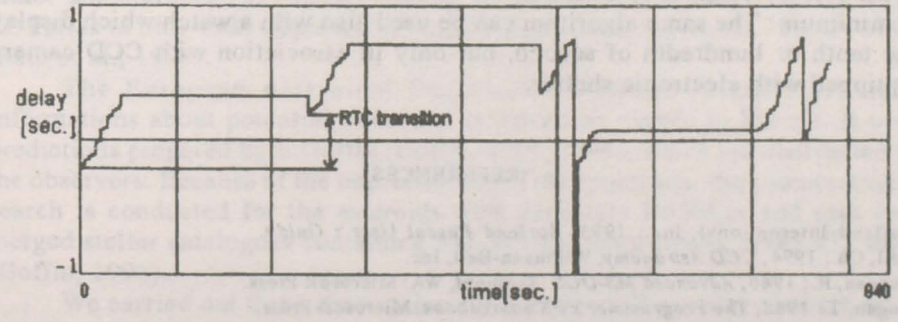



Fig. 2 – The differences (Yk–dk) + corr using the modelation. K is the number of the image.

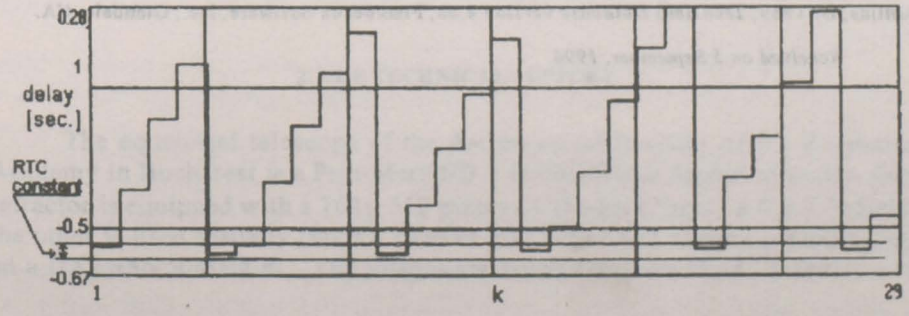



Fig. 3 – The difference of the RTC and time "photographed".

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The delay introduced by the camera is of the order of time needed to clean the CCD array, a reading of every pixel without analog to digital conversion, of about 1–2 seconds (Buil, 1994). The simplest way to correct these errors is to make the software wait until the shutter gets opened, and to perform at the same time a GET_DOS_TIME function for the image in the exposure. This implies the introduction of a DELAY(corr) instruction before the GET_DOS_TIME function in the source program, where corr. is the correction determined as we said above.

Let us say that that $d_k$ is the difference between the time recorded on the image $k$ and the time "photographed" on the same image. Let $t_k$ be the time recorded for the same image.

We can construct the sequence of the intervals elapsed between two images:

$$D_k = d_{k-1} + (t_k - t_{k-1})$$

where $D_0 = 0$. We can compute the delay using the sequence:

$$Y_k = D_k - \text{Trunc}(D_k)$$

and after that we can compare with measured values using the differences $Y_k - d_k$. These differences must be equal to a constant which represents the correction required to the CCD camera software. The plot of the sequence:

$$Y_k + \text{corr} - d_k$$

has an aspect as figure 2, where the transitions in 1 are generated when the camera recorded the change of the second on the watch display. It is obvious that the good value will give the minimum on 0, and also a series of images lead to the correct results only if the number of transitions in 1, on the plot $(Y_k - d_k, k)$, is minimum. The same algorithm can be used also with a watch which displays the tenth or hundredth of second, but only in association with CCD cameras equipped with electronic shutter.

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# REDUCTION OF ASTEROIDAL APPULSES. FIRST RESULTS IN BUCHAREST

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*Abstract:* The method presented is applied to asteroidal approaches to catalogue stars and occultations, and derives the minimum separation, the separation at conjunction and the corresponding time of these ones. Three CCD observations, reported to the European Asteroidal Occultation Network, illustrate the method.

*Key words:* Astrometry, CCD, Asteroidal Occultation, Appulse.

## 1. INTRODUCTION

Numerous papers present the opportunity to derive asteroidal diameters from stellar occultations (Millis and Elliot 1979; Millis and Dunham, 1989). The first endeavours to observe the occultation by the planet (3) Juno was recorded in Sweden, in February, 1958. It was followed by a photoelectric observation in India, involving (2) Pallas (Taylor, 1962). The most observed minor planets were (2) Pallas, (3) Juno, (9) Metis and the best observed were (2) Pallas in May 1983 (Dunham et al., 1990) and (381) Myrrha in January 1991 (Sato et al., 1993).

The European Asteroidal Occultation Network (EAON) provides informations about potential asteroidal occultations visible in Europe. It uses predictions prepared by E.Goffin (Goffin, 1994, 1995), which are distributed to the observers. Because of the improbability of the ephemeris, the computational search is conducted for the asteroids with diameters D>30Km and uses two merged stellar catalogues containing 531,315 and respectively 5,487,472 stars (Goffin, 1995).

We carried out three observations in Bucharest Observatory (which were proved as being approaches); the results of first of these were included in the EAON Informations (Delhaye and Boninsegna, 1996).

## 2. THE TECHNICAL SUPPORT

The equatorial telescope of the Astronomical Institute of the Romanian Academy in Bucharest is a Prin-Merz f/D = 6000/380mm double refractor. One refractor is equipped with a 768 x 512 pixels CCD which "sees" a 4' x 2.'6 field; the other is used visually (with a field of 20'). The CCD images are supported on a DX2 486/50MHz PC. The images are taken using the "Zap" software and



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are measured using "Astrol v.3.0" (Colas and Serrau, 1991). The software records the UT, ensured by a Rhode und Schwarz time-keeper which controls the computer (Vass et al., 1996). The reduction of the data and the analysis of the phenomena are carried out by a package written in Turbo Pascal which runs on the PC.

## 3. THE METHOD OF REDUCTION

It gets relative coordinates in the instantaneous system of coordinates related to the camera. We agree to note these coordinates by $(X, Y)$. Because of the small variations in driving the instrument, the $(X_s, Y_s)$ coordinates of the stars related to the CCD system vary throughout the time of the observations (which could last two hours). Consequently, if we denote by $(X_a, Y_a)$ the relative coordinates of the asteroid in the CCD system, then the coordinates of the asteroid relative to the star are:

$$X = X_a - X_s, \quad Y = Y_a - Y_s \tag{1}$$

The determination of the moment of minimum separation and the value of that is performed by using two analyses.

The first one considers $Y$ as a function of $X$:

$$Y = Y(X) \tag{2}$$

its expressions are set up using a classical least squares method on the set of observations.

Using a 0.001 step in $X$ in the equation (2), we compute the separation star-planet in pixels (which are converted in arcseconds by multiplying with the instrumental factor of 0.618786) and we get the coordinates corresponding to the minimum approach.

At the same time, let us consider $X$ and $Y$ as linear functions of time, determined in the same manner as (2):

$$X = X(t), \quad Y = Y(t) \tag{3}$$

Introducing in these equations the values of $X$ and $Y$, corresponding to the minimum distance between the asteroid and the star, we can solve the two resulting equations and obtain the moment of the approach.

The second analysis performs the transformation star-CCD field, using the classical relations between the equatorial and ideal (standard) coordinates (Smart, 1948).

A preliminary determination of the camera orientation must be done, using another image, with a sufficient number of reference (catalogue) stars (Vass et al., 1996).

The simplest method to compute the coefficients of the orientation is the observation of a close catalogue couple in the neighbourhood of the asteroid. We found such couples of stars by using a star maps software (Văduvescu and Bîrlan, 1996).

The moment and the value of the minimum separation can be extracted in a similar way as in $X$ and $Y$, using equatorial coordinates. Additionally, we compute the conjunction in equatorial coordinates and the corresponding time.

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The analysis in equatorial coordinates makes possible the reduction of the positions to geocenter; the use of geocentric positions allows us to compare our results with the predictions, and with the results of other observers.

## 4. FIRST RESULTS

On August 28/29, 1995, within an hour, we took 80 images of (7) Iris ($V = 9.06$) with 2s time of exposure, in the neighbourhood of PPM 92838 ($V = 9.6$). The analysis of the positions by both methods derived the same separation (without occultation) for Bucharest (Văduvescu, 1996). Figure 1 plots the 80 topocentric positions of (7) Iris and of the star.

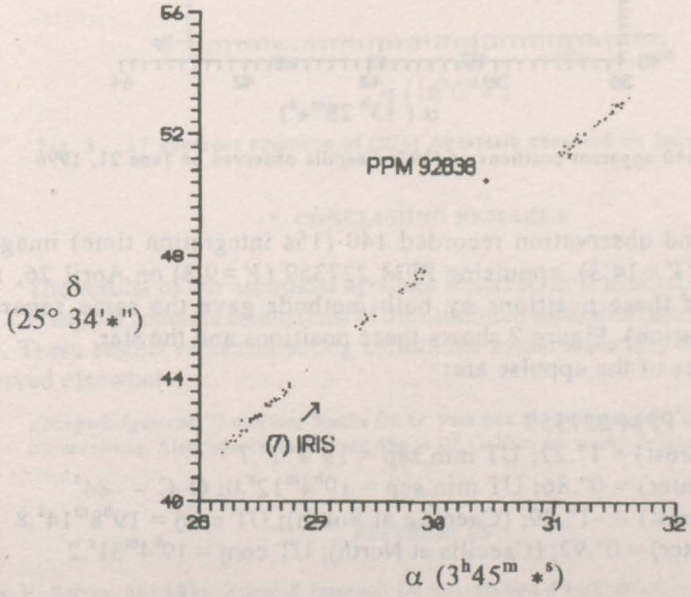



Fig. 1 – 80 apparent positions of (7) Iris observed on August 28, 1995

The results of this phenomenon (O–C are related to the predictions of E.Goffin, distributed by EAON) are:

(7) Iris – PPM 92838
Min.sep (Bucharest) = 1".20; UT min.sep = $0^h17^m44^s.8$
Min.sep (geocenter) = 4".20; UT min.sep = $0^h19^m45^s.5$; O–C = $-32^s$
$\Delta\delta$ conj (Bucharest) = –1".24; (Iris at South); UT conj = $0^h17^m26^s.9$
$\Delta\delta$ conj (geocenter) = 4".22; (Iris at North); UT conj = $0^h20^m46^s.0$

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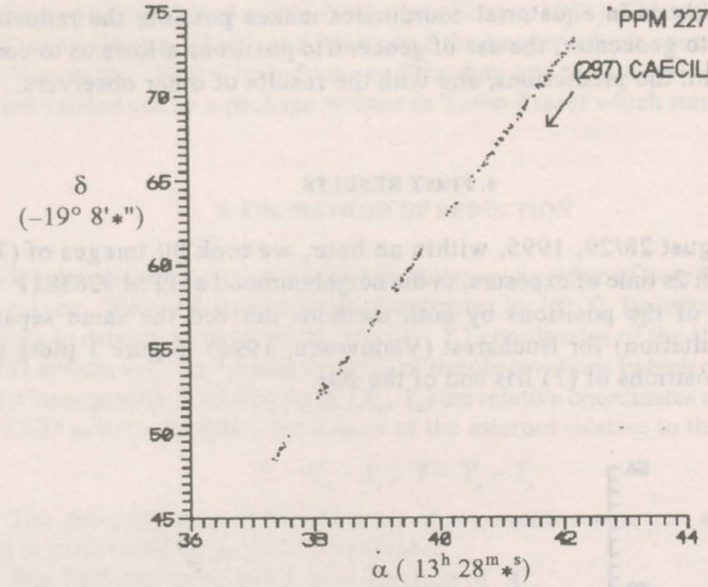



Fig. 2 – 140 apparent positions of (297) Caecilia observed on June 21, 1996

The second observation recorded 140 (15s integration time) images of (297) Caecilia ($V$ =14.3), appulsing PPM 227359 ($V$ =9.8) on April 26, 1996. The analysis of these positions by both methods gave the same separation (without occultation). Figure 2 shows these positions and the star.

The values of the appulse are:

(297) Caecilia – PPM 227359
Min.sep (Bucharest) = 1".22; UT min.sep = $19^h9^m7^s.7$
Min.sep (geocenter) = 0".86; UT min.sep = $19^h4^m12^s.0$; O–C = $-84^s$
Δδ conj (Bucharest) = –1".29; (Caecilia at South); UT conj = $19^h8^m14^s.8$
Δδ conj (geocenter) = 0".92; (Caecilia at North); UT conj = $19^h4^m51^s.2$

The third observation was recorded on June 21, 1996 and represents the appulse of PPM 706340 ($V$ =9.3) by (824) Anastasia ($V$ =14.07) (Goffin, 1995). Because of a technical pro-blem, only 17 CCD images were recorded within $2^h.5$, using 10s and 20s times of integration. Visually, no extinction in the star luminosity was observed between UT $20^h45^m$ and $21^h15^m$.

The apparent positions, plotted in figure 3, gave the following results:

(824) Anastasia – PPM 706340
Min.sep (Bucharest) = 1".34; UT min.sep = $21^h10^m43^s.1$
Min.sep (geocenter) = 2".25; UT min.sep = $21^h16^m31^s.8$; O–C = $10^s$
Δδ conj (Bucharest) = –1".40; (Iris at South); UT conj = $21^h11^m45^s.9$
Δδ conj (geocenter) = 2".35; (Iris at South); UT conj = $21^h14^m36^s.0$

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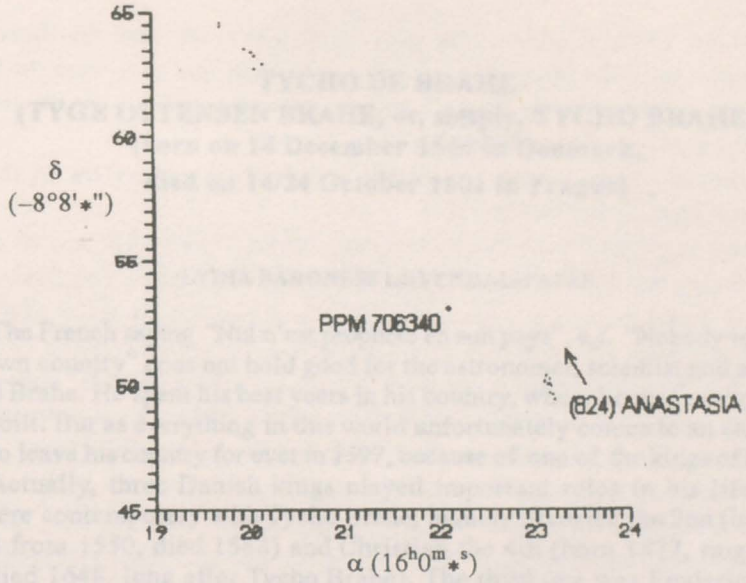



Fig. 3 – 17 apparent positions of (824) Anastasia observed on June 21, 1996.

## 5. CONCLUDING REMARKS

The results of the asteroidal appulses observed in Bucharest Observatory with the CCD supply accurate estimations of the minimum separation and the corresponding time. These results represent strong constraints in the reduction of the occultation observed elsewhere.

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# TYCHO DE BRAHE
## (TYGE OTTENSEN BRAHE, or, simply, TYCHO BRAHE)
## (born on 14 December 1546 in Denmark, died on 14/24 October 1601 in Prague)

LYDIA BARONESS LØVENDAL-PAPAE

The French saying "Nul n'est prophète en son pays", e.i. "Nobody is a prophet in his own country" does not hold good for the astronomer, scientist and artist Tyge (Tycho) Brahe. He spent his best years in his country, where he was recognized and even spoilt. But as everything in this world unfortunately comes to an end, he was forced to leave his country for ever in 1597, because of one of the kings of Denmark.

Actually, three Danish kings played important roles in his life. Two of them were contemporary with Tycho Brahe, namely Fredrick the 2nd (born 1534, reigned from 1550, died 1588) and Christian the 4th (born 1577, reigned from 1588, died 1648, long after Tycho Brahe). The third one was Frederick the 3rd (born 1609, after Tycho Brahe, reigned from 1648, died 1670). The first one made Brahe's life heaven, the second one made it hell, and the third one bought a great part of his work, which had been left in Prague and which has been ever since in the Royal Library in Copenhagen, also called the "Kongelige Bibliotek".

Much has been written about the life of Tycho Brahe (as he was called abroad). I translated many pages from German, French and especially Danish documents.

I present these data in a double position, namely both as Tycho Brahe's genealogist and blood relation.

As a genealogist I have studied his ancient aristocratic origin from Halland and Skaane, regions which today are part of the far south of Sweden and for which the Danes fought many times with the Swedes, in spite of the fact that both these peoples had the same origin! (One has heard of similar cases both in the old and in the contemporary histories... Although all people are related not only from Adam's time, as the Bible says, but also of late, it seems that our contemporaries do not know their history and even less their genealogy. This is a pity, because if they knew them, then maybe the great weapon producers would not have who to sell their weapons to and our planet would finally be able to turn around its axis and around the Sun more contentedly. It comes to our mind a saying of our Latin ancestors, namely "Inter arma silent musae", which in this case would mean "between weapons the muses ...of culture are silenced").

Before the tables of Tycho Brahe's genealogy and of our kinship, I shall make a small part of a short commentary on Tycho Brahe, edited by the Press Office of the Ministry of Foreign Affairs in Copenhagen. I quote:

"Tycho Brahe's researches embraced all fields of astronomy: the position of fixed stars, the paths of the Sun, the Moon and of the planets, as well as the determination of the comets. To all this the famous discovery of a new star, in the evening of



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11 November 1572, is added. His most important researches are those on the planets, especially the planet Mars, owing to the fact that they were the basis on which Kepler, who became Brahe's collaborator in Prague, built his renowned laws of the motion of the planets".

I should like to mention also that one of the great craters on the Moon was named after Tycho Brahe.

Studying who Tycho Brahe's parents were, I was surprised to discover that his father was first cousin to one of my ancestors, called also Brahe, whose first name was Jens, and that his mother, Beate Bille, was the sister of my great-grandmother Anne Bille, that is Tycho Brahe was my blood relation twice, both through his father and mother, as well as through all his parents' ancestors.

Similarly, the aforementioned three kings of Denmark who had close relations with Tycho Brahe, the first two with his life and the third one with the preservation of his astronomical message, were direct ancestors of all barons and counts of Løvendal and willy-nilly of the author of these lines....this has been of no benefit to us, as you can easily imagine what we went through until recently.

My father, the many-sided artist Baron George Løvendal, was the last male descedant of the Løvendals. I am the last female descendant of our family. There is also my nephew Ariadna Ruxandra who, although does not bear the title of Baroness of Løvendal, has it nevertheless in her blood. (She is the daughter of my departed sister Irene.)

Tycho Brahe was both a poet and a patriot. Had he not died too soon, he would have certainly returned to Denmark. Away from his country, he wrote a poem in Latin, which was translated by the poet J. L. Heiberg in Danish. I found a fragment of it in "Dansk Biografisk Leksikon", i.e. "The Danish Biographical Dictionary", which I translated in Romanian in prose, and my husband, the poet and architect Radu Mihai Papae, rewrote it in verse. Translated into English it is:

Fragment from

ELEGY TO DENMARK

Tell me my country which is my fault,
for which you have taken me out of your heart ?
Am I not your son, the light which embraces you in its love ?
Am I not the one who raised to the stars and glory
and honour your destiny ?
Let the sky be the gift of my love to you, a divine witness.

Here are the genealogical tables:

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## TYCHO BRAHE

**Genealogical table**

Family BRAHE originated from
**Halland and Skaane**
(Denmark, today Sweden)[1]

♡*VERNER BRAHE (c.XIII)
♡*NIELS VERNERSEN BRAHE (c.XIII–XIV)
♡*...NIELSSEN BRAHE (first name unknown)
♡*PEDER BRAHE (mentioned in 1364 and 1376)
♡*AXEL PEDERSEN BRAHE (mentioned in 1398 and 1495)
♡*PEDER AXELSEN BRAHE (mentioned in 1441)
♡* Berette Bondesdatter Thott
♡*AXEL PEDERSEN BRAHE (died in 1487)
x♡* Maren Tygesdatter Lunge

♡ AXEL AXELSEN BRAHE (d.1551) *TYGE AXELSEN BRAHE (d.1523)
← brothers →
x 1° Anne Laugesdatter Brock — x 1° Magdalene Olufsd.Krognos
x 2° Sophie Holgersd.Rosenkrantz — x 2°* Sophie Jørgensd.Rud.
♡ JENS AXELSEN BRAHE (d.1561) — * OTTE TYGESEN BRAHE (1518–71)
← cousins →
x ♡ Anne Clausensdatter Bille — x * Beate Clausensdatter Bille
← sisters →
♡ PEDER JENSSEN BRAHE (d.1610) — *** TYGE (TYCHO) OTTESEN BRAHE (1546–1601)**
← second cousins (father) →
← first cousins (mother) →
x♡Margrethe Albrechtsd.Gjøe — x Kirstine Barbara (unofficial wife)

♡METTE PEDERSDATTER BRAHE (1579–1641)
x♡Otte Jørgenssen Marsvin

♡MARGRETHE OTTESDATTER MARSVIN (1606–1650)
x♡Jorgen Knudsen Urne

♡SOPHIE JØRGENSDATTER URNE (1629–1714)
x♡Ulrik Frederik Gyldenløve (1638–1704), Vice-King of Norway, first son of the King of Denmark and Norway ♡FREDERIK III and of

♡Margrethe von Pape, first baronesse **Løvendal**. They are the founders of all the barons and counts of Løvendal!

[1] ♡ – relatives with the author
* – relatives with the astronomer
d – deceased

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♡* **VERNER BRAHE** ( 13th century), the oldest ancestor mentioned in the tree of the noble family of the BRAHES, by origin from Halland and Skaane. He was also one of the direct ancestors of the LØVENDAL barons and counts.

♡* **NIELS VERNERSEN BRAHE** of Gyllebo (13th–14th centuries), the son of the former, mentioned as the grandfather of Peder B.

♡* ..... a son whose name is no longer known, (14th century).

♡* **PEDER BRAHE** (mentioned in 1364 and 1376), the acknowledged ancestor of the Brahe family, as well as of the Løvendal counts and barons.

♡* **AXEL PEDERSEN BRAHE** from Vidskøfle (mentioned in 1398 and 1425), common ancestor of the Brahe family as well as of the Løvendal counts and barons. (His wife is not mentioned!)

♡* **PEDERSEN AXEL BRAHE** of Vidskøfle (mentioned in 1441). Before 1442 married to BERETTE BONDESDATTER THOTT; d. 26 July, 1474 (married for the second time to Follert van Knobe, who died in 1452; married for the third time, approx. in 1454, to Johan Pedersen Oxe, from Tordsø and Viksø and d.approx. in 1490). Parents: ♡* BONDE JEPSEN THOTT, d.approx. 1440 and ♡* SIDSEL NIELSDATTER JERNSKAEG, also direct ancestors of the Løvendal barons and counts.

♡* **AXEL PEDERSEN BRAHE** from Krogholm and Tostrup (Ingelstad), (d.1487). Married to ♡*MAREN TYGESDATTER LUNGE, deceased at the beginning of 1520. They were direct ancestors also of the Løvendal barons and counts.

* **TYGE AXELSEN BRAHE** from Tostrup, knight and state counsellor, dead in battle nearby Malmø, in 1523. Married for the first time, in 1503, to MAGDALENE OLUFSDATTER KROGNOS, d.4 September, 1510; married for the second time, in approx. 1513, to *SOPHIE JØRGENSDATTER RUD, d.1 March, 1555, at Tureby (she married for the second time, before 22 July, 1528, to Erik Bolle, d.1562). Parents: *JORGEN MIKELSEN RUD, d.1504–1505 and * KIRSTINE ERIKSDATTER ROSENKRANTZ, d.1509. (Tyge Axelsen Brahe's brother,

♡* **AXEL AXELSEN BRAHE**, was a direct ancestor of the Løvendal barons and counts.).

* **OTTE TYGESEN BRAHE** from Knudstrup, state counsellor, marshal, born on 2 October, 1518, in Tostrup, dead on 9 May, 1591, at Helsingborg castle; he was also the founder of the older branch of the newer Brahe family. Married on 23 November, 1544, to *BEATE CLAUSDATTER BILLE, born on 29 April, 1526, at Skarholt, in Skaane, dead on 18 October, 1605. Parents: *CLAUS STEENSEN BASSE BILLE, born in approx. 1490, dead on 4 January, 1558 and * LISBET JENSDATTER ULFSTAND, d.5 April, 1540 (Beate was ANNE BILLE's sister, who was an ancestor of the Løvendal barons and counts; Anne's husband, JENS AXELSEN BRAHE, d.1561, was first cousin of this Otte Tygesen Brahe!).

* **TYGE (TYCHO) OTTESEN BRAHE**, the famous astronomer, was born on 14 December, 1546 at Knudstrup, in Skaane and died on 14/24 October, 1604, in Prague. In 1573 he was living in a free union with Kirstine Barbara, deceased in 1604, in Prague and buried in the same family tomb as Tycho Brahe. (Tycho Brahe was second cousin of PEDER JENSEN BRAHE, d.1610, the direct ancestor of the Løvendals, whose mother, ANNE CLAUSDATTER SILLE, was the sister of Tycho's mother, that is his aunt; consequently, her children were first cousins of the great astronomer!)

TYGE (TYCHO) BRAHE and Kirstine Barbara's children did not bear the name of BRAHE, being illegitimate, but the descendants of his astronomical activity spread all over our earthly world.

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**URIEL FRISCH – *Turbulence: The Legacy of A. N. Kolmogorov*, Cambridge University Press, 1995, ISBN 0–521–45713–5, paperback, £ 15.95, US $ 29.95.**

The author of this book - Uriel Frisch - works at the Observatoire de la Côte d'Azur, but also has the opportunity to held lectures for graduate students in "Turbulence and Dynamical Systems" at the University of Nice-Sophia-Antipolis. Based on these lectures and helped by the questions that appeared during teaching this material was born this book.

One goal of it was to revisit Kolmogorov's work from 1941 and to bring forward all its ideas. It is supposed that the readers have some knowledge of fluid mechanics, but are at the beginning with the modern theory of probabilities. We also note that concepts very important in dynamical systems studies (such as symmetry-breaking and deterministic chaos) are also met here, and there is more emphasis on understanding them than on mathematical rigor.

The introductory chapter about turbulence and symmetries is followed by a study of Navier-Stokes equations for incompressible fluids. Then, it is an attempt to ask the question if the probabilistic description is appropriate for turbulence. After this there are presented some tools from probabilistic theory used in turbulence and then the basic experimental laws of fully developed turbulence that provide Kolmogorov's theory from 1941 (chapter 5).

This theory is exposed in detail in chapter 6 in a very personal manner. Then it is discussed the phenomenology of turbulence (chapter 7) and the intermittency (chapter 8). At the end, in chapter 9, there is a guided tour on further reading on turbulence.

As we can see from this brief presentation of the book it reaches many hot and interesting subjects of today's turbulence studies.

It is useful to those who want to learn the basics of turbulence or to have a fresh look of this subject. Therefore, the readers could be students in the first years of graduate studies or researchers in astrophysics, mathematics, physics or geophysics because this 'fully developed turbulence' is equally met in cosmical and natural environments.

*Cristina Blaga*

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# INTERNATIONAL ASSOCIATION "ECLIPSA '99"

On August 11, 1999 a total solar eclipse will occur, whose maximum will be in Romania.

There will be the Sun's maximum coverage (103%), maximum duration (2min 23s) and observation height (on mountain tops of over 2,500m height) as well as the best observation time (11h UT). On the totality band, which has a width of 112km, there is only one major astronomical observatory, that in Bucharest, just on the central line!

It is the only total eclipse visible in Europe for the next three decades.

The Romanian astronomers have already started the preparations for this event from both scientific and public standpoints.

We have the pleasure to announce that the *International Association Eclipsa '99* – **"E'99"** – was founded.

Supplementary informations concerning its activity and possibility to become an **"E'99"** member will be provided at request.

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Manuscripts should be submitted in dupplicate; they must be typewritten on white A4–sized paper, onesided and doublespaced (enclosing abstract, references, footnotes and figure captions). The first page should contain: the article's title (brief and informative), author's name and affiliation, followed by an Abstract in English and Key words. The text should be clear and concise (it is recommended not to exceed 10 pages). The Abstract will present clearly the principal conclusions of the work, in no more than 10–15 lines.

*Chapters and Paragraphs*. Papers, except short notes, should be divided into Chapters, numered by Arabic numerals. Chapter may be divided into Paragraphs denoted by the number of the Chapter and the number of Paragraph; each Chapter and each Paragraph should have a short descriptive title (e.g. "3.2. Results").

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*The Places* of tables and figures within the text have to be marked with lead pencil on the left margin of the manuscript.

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