ROMANIAN ACADEMY

ROMANIAN ASTRONOMICAL JOURNAL

Vol. 7, No. 2
1997

EDITURA ACADEMIEI ROMÂNE

ROMANIAN ACADEMY

ROMANIAN ASTRONOMICAL JOURNAL

The ROMANIAN ASTRONOMICAL JOURNAL appears twice a year.
Orders from abroad (issues or subscriptions) should be sent to:
RODIPET S.A. Piaţa Presei Libere nr. 1, P. O. Box 33–57, Fax: 401-222 6407, Telephone: 401-618 5103, 401-222 4126, Bucureşti, România;
ORION PRESS INTERNATIONAL SRL, Şos. Olteniţei, 35–37, Sect. 4, P.O. Box 61–170 Bucharest 4, Fax: 401-312 2425, 401-634 7145, Telephone: 401-634 6345, Bucureşti, România;
EDITURA ACADEMIEI ROMÂNE, Calea 13 Septembrie, nr. 13, sector 5, P.O. Box 5–42, Bucureşti, România, RO-76117, Tel. 401-411 9008, Tel./Fax. 401-410 3983; 401-410 3448.

The manuscripts, the books and journals proposed in exchange and the mail should be sent to the Editorial Board.

Editorial Board's Address:

ROMANIAN ASTRONOMICAL JOURNAL
Institutul Astronomic,
Str. Cuţitul de Argint 5,
Cod 75212, Bucureşti 28,
Fax: (401) 337 33 89,
Telephone: (401) 335 68 92,
ROMANIA

Calea 13 Septembrie 13, sector 5, Bucureşti, tel. 410.38.46

ROMANIAN ASTRONOMICAL JOURNAL

Vol. 7, No. 2, 1997

CONTENTS

Rom. Astrom. J., Vol. 7, No. 2, p. 119–208, Bucharest, 1997

SPATIALLY FLAT ROBERTSON–WALKER COSMOLOGY WITH MIXED RADIATION AND THERMALIZED MASSLESS SCALAR FIELD CONTENT

CIPRIAN DARIESCU[1], SHINJI HAMAMOTO[2]

[1] *Department of Theoretical Physics "Al.I. Cuza" University, 6600 Iaşi, Romania*

[2] *Department of Physics, Toyama University, Gofuku, Toyama 930, Japan*

Abstract. Basically, in this paper, our interest has been focused on the following problem: how did the thermalization of the minimally coupled massless scalar field go in the early radiation dominated spatially flat Robertson–Walker Universe and what kind of cosmological consequences did it have by its back-reaction effect? Therefore, we have worked out explicitly the *exact* frequency modes, quantized the massless field on the $(k = 0)$-radiation dominated background and, finally, have estimated the *many-particle* expectation value of the conservative energy-momentum tensor, assuming a Bose-Einstein distribution (at the radiation equilibrium temperature) of the quanta. Unlike the Minkowskian case, when $w = 3p$ for the massless scalar source, this time, we get equal supplementary contributions, $\dfrac{h_0^2 T^2}{24a^4}$, to the pressure and the energy-density of the thermalized field. They mainly come from the decaying amplitude of the exact frequency modes and principially, strongly influence the evolution of the mixed radiation and thermalized massless scalar field dominated $(k = 0)$-Robertson–Walker Universe by the back reaction effect. It arises a maximal, Planckian-like, critical temperature $3/\sqrt{\pi}$ (in Planck units) and the mixed matter Universe smoothly goes from the phase of the coherent massless scalar field domination, when $a \approx t^{1/3}$ and the deceleration parameter $q = 2$, to the usual phase of radiation $a \approx t^{1/2}$ and $q = 1$. The exact solution of Einstein equations over the whole of this transition has been concretely estimated in closed form as well as the "luminosity-distance versus cosmological red-shift" relation that fully characterizes the model.

Key words: spatially flat Robertson–Walker Universe, thermal background radiation, scalar field quantization, thermalization procedure, cosmologies with mixed matter content.

1. INTRODUCTION

As it is known, the ($k = 0$)-Robertson–Walker Universe is generically described by the line element

$$ds^2 = e^{2f(t)}\delta_{\mu\nu}dx^\mu dx^\nu - (dt)^2, \ \mu, \nu = 1, 2, 3 \tag{1}$$

with $t = x^4$ being the cosmic (universal) time and $f(t)$ the primitive of the Hubble function $h(t)$. With respect to the pseudo-orthonormal tetrads (dual to each other)

$$\begin{aligned} &\text{a)} \quad e_\mu = e^{-f}\partial_\mu, \ e_4 = \partial_4 \\ &\text{b)} \quad \omega^\mu = e^f dx^\mu, \ \omega^4 = dx^4 \end{aligned} \tag{2}$$

the only non-vanishing components of the Einstein tensor read

$$\begin{aligned} &\text{a)} \quad G_{\mu\nu} = -\left[2f_{,44} + 3(f_{,4})^2\right]\delta_{\mu\nu} \\ &\text{b)} \quad G_{44} = 3(f_{,4})^2, \text{ with } f_{,4} = e_4(f) = \frac{df}{dt} \end{aligned} \tag{3}$$

and the Einstein's field equations with a "background radiation" content,

$$\begin{aligned} &\text{a)} \quad T^{(r)}_{\mu\nu} = \frac{1}{3}w_3\delta_{\mu\nu} \\ &\text{b)} \quad T^{(r)}_{44} = w_r, \end{aligned} \tag{4}$$

where

$$w_r = \frac{\pi^2}{15}T_r^4 \tag{5}$$

is the energy density at the radiation equilibrium temperature T_r, have the explicit form:

$$\begin{aligned} &\text{a)} \quad 2f_{,44} + 3(f_{,4})^2 = -\frac{\kappa_0\pi^2}{45}T_r^4 \\ &\text{b)} \quad 3(f_{,4})^2 = \frac{\kappa_0\pi^2}{15}T_r^4 \end{aligned} \tag{6}$$

The differential equation of the scale function f is readily given by the combination $\frac{1}{6}[3(6.a) + (6.b)]$, namely

$$f_{,44} + 2(f_{,4})^2 = 0 \tag{7}$$

and possesses the non-trivial solution

$$f(t) = \frac{1}{2}\ln(\alpha t) \tag{8}$$

in the usual Minkowskian-like calibration at $t = \alpha^{-1}$:

$$f(\alpha^{-1}) = 0; \ \ h(\alpha^{-1}) \equiv h_0 = \frac{1}{2}\alpha \tag{9}$$

Hence, the line element (1) and the (radiation) temperature evolution law, $T_r(t)$, are concretely given by (Robertson and Noonan 1968, Weinberg 1972)

$$\begin{aligned} &\text{a)} \quad ds^2 = (2h_0 t)\delta_{\mu\nu} dx^\mu dx^\nu - (dt)^2 \\ &\text{b)} \quad T_r(t) = \left(\frac{45}{4\kappa_0}\right)^{1/4} (\pi t)^{-1/2} \end{aligned} \tag{10}$$

2. THE MASSLESS SCALAR SOURCE AND ITS EXACT FREQUENCY MODES

Now, let us pass to the case when a minimally coupled massless scalar field ϕ, which later will be assumed in thermodynamical equilibrium with itself at a temperature T_s, comes into play. With respect to (2), its Lagrangian

$$L[\phi] = \frac{1}{2}\eta^{ab}\phi_{|a}\phi_{|b} \tag{11}$$

leads to the energy-momentum tensor

$$T^{(s)}_{ab} = \phi_{|a}\phi_{|b} - \eta_{ab}L[\phi] \tag{12}$$

and to the Gordon equation

$$\Delta\phi - (2h_0 t)\left[\phi_{,44} + 3h_0(2h_0 t)^{-1}\phi_{,t}\right] = 0, \quad \text{with } \Delta = \delta^{\mu\nu}\partial^2_{\mu\nu} \tag{13}$$

on the curved background (10.a). Due to the flatness of the $\{t = const\}$-hypersurfaces, the eigenmodes, $u_{\boldsymbol{k}}(x)$, that appear in the usual (generalized) Fourier representation

$$\phi(x) = \int d^3k \left[a_{\boldsymbol{k}} u_{\boldsymbol{k}}(x) + a^+_{\boldsymbol{k}} \overline{u}_{\boldsymbol{k}}(x)\right] \tag{14}$$

of the field operator ϕ and satisfy the orthonormalization relation

$$(u_{\boldsymbol{k}'}, u_{\boldsymbol{k}}) = i\int \left[\overline{u}_{\boldsymbol{k}'} u_{\boldsymbol{k},4} - \overline{u}_{\boldsymbol{k}',4} u_{\boldsymbol{k}}\right](2h_0 t)^{2/3} d^3x = \delta\left(\boldsymbol{k}' - \boldsymbol{k}\right), \tag{15}$$

can be decomposed in

$$\begin{aligned} &\text{a)} \quad u_{\boldsymbol{k}}(x) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}} T^{(+)}_{\boldsymbol{k}}(t) \\ &\qquad\qquad\qquad\qquad , \text{ with } \overline{T}^{(+)}_{\boldsymbol{k}}(t) = T^{(-)}_{\boldsymbol{k}}(t) \\ &\text{b)} \quad \overline{u}_{\boldsymbol{k}}(x) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}} T^{(-)}_{\boldsymbol{k}}(t) \end{aligned} \tag{16}$$

Thus, with (16) and (14), the Gordon equation (13) provides the differential equation

$$\frac{d^2T_k}{dt^2} + \frac{3/2}{t}\frac{dT_k}{dt} + \frac{|k|}{2h_0 t}T_k = 0 \tag{17}$$

for the time-dependent functions $T_k^{(\pm)}(t)$. Performing the coordinate transformation

$$2h_0 t = \eta^2, \quad \text{with } t, \eta \in \mathcal{R}_+ \tag{18}$$

and the (function) substitution

$$T_k(t) = \eta^\gamma S'_k(\eta), \quad \text{with } \gamma \in \mathcal{R}, \tag{19}$$

one gets from (17),

$$\frac{d^2S_k}{d\eta^2} + \frac{2}{\eta}(\gamma+1)\frac{dS_k}{dt} + \left[|\kappa|^2 + \frac{\gamma(\gamma+1)}{\eta^2}\right]S_k = 0, \quad \text{with } k = \frac{k}{h_0}. \tag{20}$$

For $\gamma = -1$, it obviously comes to the well-known differential equation

$$\frac{d^2S_k}{d\eta^2} + |k|^2 S_k = 0, \tag{21}$$

possessing the fundamental solutions

$$S_k^{(\pm)}(\eta) = N_k e^{\mp ik\eta}, \tag{22}$$

where, the normalization constant N_k will be appropriately determined from (15). Hence, going back along the chain (22), (19), (18), the eigenmodes $\{u_k(x), \overline{u}_k(x)\}$ are explicitly given by

$$\text{a)} \qquad u_k(x) = \left(32\pi^3|k|h_0 t\right)^{-1/2} \exp\left[i\left(k\cdot x - \frac{|k|}{h_0}\sqrt{2h_0 t}\right)\right]$$

$$\text{b)} \qquad \overline{u}_k(x) = \left(32\pi^3|k|h_0 t\right)^{-1/2} \exp\left[-i\left(k\cdot x - \frac{|k|}{h_0}\sqrt{2h_0 t}\right)\right] \tag{23}$$

with (23.a) and (23.b) for positive and negative frequencies, respectively.

3. CANONICAL QUANTIZATION AND THERMALIZATION PROCEDURE

As the canonical momentum π of ϕ can be obtained from $L[\phi] = -\sqrt{g}L[\phi]$ by the usual definition

$$\pi = \frac{\partial}{\partial\phi_{,4}}\left[-\sqrt{-g}L[\phi]\right] = \sqrt{-g}\phi_{,4} \tag{24}$$

and concretely reads

$$\pi(\boldsymbol{x},t) = -i\left[\frac{2h_0t}{(2\pi)^3}\right]^{1/2}\int\frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{2|\boldsymbol{k}|}}\left\{\left[|\boldsymbol{k}| - \frac{ih_0}{\sqrt{2h_0t}}\right]a_{\boldsymbol{k}}\mathrm{e}^{i\left(\boldsymbol{k}\cdot\boldsymbol{x}-\frac{k}{h_0}\sqrt{2h_0t}\right)} - \right.$$
$$\left. - \left[|\boldsymbol{k}| + \frac{ih_0}{\sqrt{2h_0t}}\right]a^+_{\boldsymbol{k}}\mathrm{e}^{-i\left(\boldsymbol{k}\cdot\boldsymbol{x}-\frac{k}{h_0}\sqrt{2h_0t}\right)}\right\} \tag{25}$$

for

$$\phi(\boldsymbol{x},t) = \frac{1}{\sqrt{16\pi^3h_0t}}\int\frac{\mathrm{d}^3k}{\sqrt{2|\boldsymbol{k}|}}\left\{a_{\boldsymbol{k}}\mathrm{e}^{i\left(\boldsymbol{k}\cdot\boldsymbol{x}-\frac{k}{h_0}\sqrt{2h_0t}\right)} + a^+_{\boldsymbol{k}}\mathrm{e}^{-i\left(\boldsymbol{k}\cdot\boldsymbol{x}-\frac{k}{h_0}\sqrt{2h_0t}\right)}\right\} \tag{26}$$

one gets, imposing the "equal time" Bohr-Heisenberg quantization relation

$$[\pi(\boldsymbol{x}',t),\phi(\boldsymbol{x}',t)] = -i\delta(\boldsymbol{x}'-\boldsymbol{x}), \tag{27}$$

the well-known algebra

$$\begin{aligned} &\text{a)}\ [a_{\boldsymbol{k}}, a_{\boldsymbol{k}}] = \left[a^+_{\boldsymbol{k}}, a^+_{\boldsymbol{k}}\right] = 0 \\ &\text{b)}\ \left[a_{\boldsymbol{k}}, a^+_{\boldsymbol{k}}\right] = \delta(\boldsymbol{k}'-\boldsymbol{k}), \end{aligned} \tag{28}$$

satisfied by the annihilation and creation operators $a_{\boldsymbol{k}}$, $a^+_{\boldsymbol{k}}$ (Birell and Davies 1982, Kaku 1993). Thus, in principle, the normal ordered Lagrangian and energy-momentum tensor operators of the field ϕ have the expressions

$$\begin{aligned} \text{a)}\quad :L: =& \int a^+_{\boldsymbol{k}'}a_{\boldsymbol{k}}\eta^{ab}\overline{u}_{\boldsymbol{k}'|a}u_{\boldsymbol{k}|b}\mathrm{d}^3k'\mathrm{d}^3k + \\ &+ \frac{1}{2}\eta^{ab}\int\left[a_{\boldsymbol{k}'}a_{\boldsymbol{k}}u_{\boldsymbol{k}'|a}u_{\boldsymbol{k}|b} + a^+_{\boldsymbol{k}'}a^+_{\boldsymbol{k}}\overline{u}_{\boldsymbol{k}'|a}\overline{u}_{\boldsymbol{k}|b}\right]\mathrm{d}^3k'\mathrm{d}^3k \\ \text{b)}\quad :T^{(s)}_{ab}: =& \int a^+_{\boldsymbol{k}'}a_{\boldsymbol{k}}\left[\overline{u}_{\boldsymbol{k}'|a}u_{\boldsymbol{k}|b} + \overline{u}_{\boldsymbol{k}'|b}u_{\boldsymbol{k}|a}\right]\mathrm{d}^3k'\mathrm{d}^3k + \\ &+ \int\left[a_{\boldsymbol{k}'}a_{\boldsymbol{k}}u_{\boldsymbol{k}'|a}u_{\boldsymbol{k}|b} + a^+_{\boldsymbol{k}'}a^+_{\boldsymbol{k}}\overline{u}_{\boldsymbol{k}'|a}\overline{u}_{\boldsymbol{k}|b}\right]\mathrm{d}^3k'\mathrm{d}^3k - \eta_{ab}:L: \end{aligned} \tag{29}$$

and taking the expectation values for a (generic) many-particle state $|\cdots{}^i n_{\boldsymbol{k}}, \cdots\rangle$ it results

$$\begin{aligned} &\text{a)}\ \langle|:L:|\rangle = \int\mathrm{d}^3k\,\eta^{ab}\overline{u}_{\boldsymbol{k}|a}u_{\boldsymbol{k}|b}n_{\boldsymbol{k}} \\ &\text{b)}\ \langle|:T^{(s)}_{ab}:|\rangle = 2\int\mathrm{d}^3k\,\overline{u}_{\boldsymbol{k}|a}u_{\boldsymbol{k}|b}n_{\boldsymbol{k}} - \eta_{ab}\int\mathrm{d}^3k\,\eta^{cd}\overline{u}_{\boldsymbol{k}|c}u_{\boldsymbol{k}|d}n_{\boldsymbol{k}}, \end{aligned} \tag{30}$$

where $n_{\boldsymbol{k}}$ is the number density of quanta, "centered" on $\boldsymbol{k}$, in the $\boldsymbol{k}$-space (Birell and Davies 1982). For an isotropic distribution $n_{\boldsymbol{k}}$ depends on k and so, all the integrals containing free k_μ-terms vanish. Hence, the only essential components of the energy-momentum tensor (30.b) are given by

$$\begin{aligned} &\text{a) } \langle |{:}T^{(s)}_{\mu\nu}{:}|\rangle = 2\int\left[\overline{u}_{k|\mu}u_{k|\nu} - \frac{1}{2}\eta^{cd}\overline{u}_{k|c}u_{k|d}\right]n_k \mathrm{d}^3k, \text{ for } \mu=\nu \\ &\text{b) } \langle |{:}T^{(s)}_{44}{:}|\rangle = 2\int\left[\overline{u}_{k|4}u_{k|4} + \frac{1}{2}\eta^{cd}\overline{u}_{k|c}u_{k|d}\right]n_k \mathrm{d}^3k \end{aligned} \tag{31}$$

From (23) it yields immediately

$$\begin{aligned} &u_{k\mu} = \frac{ik_\mu}{\sqrt{2h_0t}}u_k;\ \overline{u}_{k\mu} = -\frac{ik_\mu}{\sqrt{2h_0t}}\overline{u}_k;\ \overline{u}_k u_k = \left(32\pi^3|\boldsymbol{k}|h_0t\right)^{-1} \\ &u_{k4} = -\frac{i}{\sqrt{2h_0t}}\left(|\boldsymbol{k}| - \frac{ih_0}{\sqrt{2h_0t}}\right)u_k;\ u_{k4} = \frac{i}{\sqrt{2h_0t}}\left(|\boldsymbol{k}| + \frac{ih_0}{\sqrt{2h_0t}}\right)u_k \end{aligned} \tag{32}$$

and consequently, passing in (31) to spherical coordinates (in the $\boldsymbol{k}$-space), $\mathrm{d}^3\boldsymbol{k} = k^2\mathrm{d}k\mathrm{d}\Omega$ with $k = |\boldsymbol{k}|$, we concretely get

$$\langle |{:}T^{(s)}_{\mu\nu}{:}|\rangle = p_s\delta_{\mu\nu};\quad \langle |{:}T^{(s)}_{44}{:}|\rangle = w_s, \tag{33}$$

where

$$\begin{aligned} &\text{a)}\qquad p_s = \frac{1}{3}\frac{D(4;n_k)}{(2h_0t)^2} + \frac{h_0^2}{2}\frac{D(2;n_k)}{(2h_0t)^3} \\ &\text{b)}\qquad w_s = \frac{D(4;n_k)}{(2h_0t)^2} + \frac{h_0^2}{2}\frac{D(2;n_k)}{(2h_0t)^3} \end{aligned} \tag{34}$$

and

$$D(a;n_k) = \int_0^\infty \frac{\mathrm{d}k}{2\pi^2}k^{a-1}n_k. \tag{35}$$

Assuming that ϕ is (effectively) thermalized at a temperature T_s, then (Kaku 1993, Linde 1990)

$$n_k = [\exp(k/T_s) - 1]^{-1} \tag{36}$$

and

$$D(a;n_k) = \frac{(a-1)!}{2\pi^2}\zeta(a)T_s^a, \tag{37}$$

where ζ is the Riemann function and thus, we get in full the pressure and the energy density of the (thermalized) massless scalar field,

$$\begin{aligned} &\text{a)}\qquad p_s = \frac{\pi^2}{90}\frac{T_s^4}{(2h_0t)^2} + \frac{h_0^2}{24}\frac{T_s^2}{(2h_0t)^3} \\ &\text{b)}\qquad w_s = \frac{\pi^2}{30}\frac{T_s^4}{(2h_0t)^2} + \frac{h_0^2}{24}\frac{T_s^2}{(2h_0t)^3} \end{aligned} \tag{38}$$

As it can be noticed, in comparison to the usual Minkowskian case when

$$p_s^{(M)} = \frac{\pi^2}{90}T_s^4;\quad w_s^{(M)} = \frac{\pi^2}{30}T_s^4, \tag{39}$$

now, there are two major differences, both with the same geometrodynamical origin, namely the cosmological expansion. The most sensitive one (to the universal expansion) is clearly reflected by the h_0^2-contribution of the second term which survives as a difference to (39) even at the moment of Minkowskian calibration, $t_0 = (2h_0t)^{-1}$. The less sensitive one, namely the explicit $(2h_0t)^{-2}$-time dependence of p_s and w_s, is due to the $(2h_0t)^{-1/2}$-decaying law of the exact modes (23) of the massless scalar field ϕ minimally coupled to the curved background of the radiation dominated ($k = 0$)-Robertson–Walker Universe. However, employing the adiabatic approximation $h_0^2 \approx 0$ (Birell and Davies 1982), it is getting clear that (39) at any instant t must equal (38), with $h_0^2 \approx 0$, which means that $T_s(t)$ from (39) has to be equal to $T_s(2h_0t)^{-1/2}$ from (38). If T_s from (38) were constant, then $T_s(t)$ would satisfy the well-known "cosmological cooling law"

$$T_s(t) = \frac{T_s^0}{a(t)}, \tag{40}$$

with $a(t) = (2h_0t)^{1/2}$ and $T_0^2 = T_s(h_0^{-1}/2)$ temperature figuring in (38). Since we claim that (33) is the energy-momentum tensor of the thermalized massless scalar field, then it has to satisfy the conservation law

$$\langle | :T^{(s)ab}: | \rangle_{;b} = 0 \tag{41}$$

namely

$$\dot{w}_s + 3\frac{\dot{a}}{a}(w_s + p_s) = 0, \tag{42}$$

which concertely comes to (employing (38))

$$\left(\frac{2\pi^2}{5}T_s^2 + \frac{h_0^2/4}{a^2}\right)\dot{T}_s = -\frac{\dot{a}}{a}T_s\left[\frac{2\pi^2}{5}(1-1)T_s^2 + \frac{3h_0^2}{4}(1-1)\frac{1}{a^2}\right] \equiv 0, \tag{43}$$

pointing out that indeed the temperature T_s figuring in (38) *is constant*, i.e. $T_s =$ $= T_s^0$, and so, the (expected) "cooling law" (40) is really valid. Hence, with respect to the effective ***time dependent*** temperature $T_s = T_s(t)$ the two expressions (38.a) and (38.b) become:

$$\begin{aligned} &\text{a)} \qquad p_s = \frac{\pi^2}{90}T_s^4(t) + \frac{h_0^2}{24}T_s^2(t)e^{-4f(t)} \\ &\text{b)} \qquad w_s = \frac{\pi^2}{30}T_s^4(t) + \frac{h_0^2}{24}T_s^2(t)e^{-4f(t)} \end{aligned} \tag{44}$$

which, at least in principle, generalizes (38) to any spatially flat RW-background of Hubble-primitive scale function $f(t)$. Now, on might say that although the temperature T_s figuring in (38) is proved by (43) to be exactly constant that does not mean actually that $T_s(t)$ in (44) drops like (40), since we have used the adiabatic approximation $h_0^2 \approx 0$ which cuts-off exactly the generalized supplementary terms (in h_0^2) appearing in (44). Therefore, we have to use again the energy evolution equation (42), for (44) this time, in order to check if $T_s(t)$ satisfies (40). It yields

$$\left(\frac{2\pi^2}{5}T_s^2 + \frac{h_0^2}{a^2}e^{-4f}\right)\dot{T}_s = -\left[\frac{2\pi^2}{5}T_s^2 + \frac{h_0^2}{4}e^{-4f}\right]T_s\dot{f}, \tag{45}$$

which clearly means

$$T_s(t) = T_s^0 e^{-f(t)} = \frac{T_0^2}{a(t)}, \quad \text{for any } f(t) \tag{46}$$

and thus, the time-dependent temperature, defining the pressure and the energy density of the massless scalar field, does really go down as the temperature of the (thermalized) background radiation.

4. THE BACK-REACTION EFFECT

With (3), (4), (33) and (44), we are now in an appropriate position to get an exact solution to the interesting cosmological model of a spatially flat Robertson–Walker Universe driven by the mixed matter content of radiation and massless scalar field in thermal equilibrium. A similar but more involved analysis for mixed radiation and thermalized *massive* scalar fields (in thermal equilibrium) can be found in (Linde 1990).

Thus, for $T_r(t) = T_s(t) = T(t)$, the essential components of the total energy-momentum tensor $T_{ab} = T_{ab}^{(r)} + \langle | : T_{ab}^{(s)} : | \rangle$ read

$$\begin{aligned} &\text{a)} \qquad T_{\mu\nu} = \left[\frac{\pi^2}{30}T^4 + \frac{h_0^2}{24}T^2 e^{-4f}\right]\delta_{\mu\nu} \\ &\text{b)} \qquad T_{44} = \frac{\pi^2}{10}T^4 + \frac{h_0^2}{24}T^2 e^{-4f} \end{aligned} \tag{47}$$

and using the non-vanishing components (3) of the Einstein tensor, we get the Einstein field equations for this model

$$\begin{aligned} &\text{a)} \qquad 2f_{,44} + 3(f_{,4})^2 = -\kappa_0\left[\frac{\pi^2}{30}T^4 + \frac{h_0^2}{24}T^2 e^{-4f}\right] \\ &\text{a)} \qquad 3(f_{,4})^2 = \kappa_0\left[\frac{\pi^2}{10}T^4 + \frac{h_0^2}{24}T^2 e^{-4f}\right] \end{aligned} \tag{48}$$

As it can be checked immediately, the multiplication of (48.a) by $3f_{,4}$ and the insertion of the time-derivative of (48.b) lead to the expected scale dependence of the temperature

$$T = T_0 e^{-f}, \tag{49}$$

such that (48.b), with the insertion (49), becomes the essential Einstein equation for the Hubble-primitive scale function f, namely

$$\bullet \qquad \frac{df}{dt} = \underline{h}_0 e^{-2f}\left[1 + \frac{\kappa_0 T_0^2}{72}\frac{h_0^2}{\underline{h}_0^2}e^{-2f}\right]^{1/2} \tag{50}$$

Since at the moment when $f = 0$ we have $T = T_0$ and $\left[\frac{df}{dt}\right]_{f=0} = h_0$, it yields from (50) that

$$h_0 = \underline{h}_0\left[1 - (T_0/T_c)^2\right]^{-1/2}, \tag{51}$$

where we have denoted by

$$\underline{h}_0 = \left(\frac{\kappa_0 \pi^2}{30}T_0^4\right)^{1/2} = 2\pi\left(\frac{\pi}{15}\right)^{1/2} T_0^2 \text{ (in Planck units)} \tag{52}$$

and by

$$T_c = (72/\kappa_0)^{1/2} = \frac{3}{\sqrt{\pi}} \text{ (in Planck units)} \tag{53}$$

The T_0^2-dependence of $\underline{h}_0$ is normal for a black body radiation content, but the presence of a sort of critical temperature (53) is somewhat new since, according to (51), the equilibrium temperature T_0, at the time of Minkowskian calibration, cannot exceed the critical (Planckian) value $3/\sqrt{\pi}$ when the corresponding Hubble constant h_0 blows up. Coming back to (50), it concretely reads

$$\frac{df}{dt} = \underline{h}_0 e^{-2f}\left[1 + \frac{(T_0/T_c)^2}{1 - (T_0/T_c)^2}e^{-2f}\right]^{1/2} \tag{54}$$

and the substitution

$$f(\chi) = \ln\left[\frac{(T_0/T_c)}{\sqrt{1 - (T_0/T_c)^2}}\sinh\chi\right], \text{ with } \chi \geqslant 0, \tag{55}$$

leads to the solution

$$t(\chi) = \frac{1}{2}\underline{h}_0^{-1}\frac{(T_0/T_c)^2}{1 - (T_0/T_c)^2}\left[\frac{1}{2}\sinh(2\chi) - \chi\right], \tag{56}$$

with the Big-Bang-like calibration $t(\chi = 0) \equiv 0$. Thus, within the frame of this χ-parametrization, the scale function $a = e^f$ is explicitly given by

$$a(\chi) = \frac{T_0}{\sqrt{T_c^2 - T_0^2}}\sinh\chi \tag{57}$$

and using the fact that

$$\frac{d\chi}{dt} = \frac{T_c^2 - T_0^2}{T_0^2} \frac{\underline{h}_0}{\sin h^2 \chi}$$

one can get the Hubble and deceleration functions, $h = \dot{a}/a$ and $q = -a\ddot{a}/(\dot{a})^2$, as:

$$\begin{aligned} &\text{a)} \quad h(\chi) = \underline{h}_0 \left[\left(\frac{T_c}{T_0} \right)^2 - 1 \right] \frac{\cos h\chi}{\sin h^3 \chi} \\ &\text{b)} \quad q(\chi) = 2 - \tan h^2 \chi \end{aligned} \tag{58}$$

From (49) and (57) it yields

$$T(\chi) = \frac{\sqrt{T_c^2 - T_0^2}}{\sin h\chi} \tag{59}$$

and the essential thermodynamical quantities are concretely expressed by

$$\begin{aligned} &\text{a)} \quad w_r = \frac{\pi^2}{15} \frac{(T_c^2 - T_0^2)^2}{\sin h^4 \chi} \\ &\text{b)} \quad p_r = \frac{\pi^2}{45} \frac{(T_c^2 - T_0^2)^2}{\sin h^4 \chi} \\ &\text{c)} \quad w_s = \frac{\pi^2}{30} \frac{(T_c^2 - T_0^2)^2}{\sin h^4 \chi} \left(1 + \frac{3}{\sin h^2 \chi} \right) \\ &\text{d)} \quad p_s = \frac{\pi^2}{90} \frac{(T_c^2 - T_0^2)^2}{\sin h^4 \chi} \left(1 + \frac{9}{\sin h^2 \chi} \right) \end{aligned} \tag{60}$$

Now, as it customary in Cosmology when one deals with a particular model, we have to stop a little bit the further general calculations and analyse in more detail the standard cosmological parameters of this model and how it matches in the asymptotic regions to some (well)-known cosmological models. So, for χ close to zero the cosmic time (56) can be expressed as

$$T = \frac{1}{3\underline{h}_0} (T_c^2/T_0^2 - 1)^{-1} \chi^3, \tag{61}$$

the Hubble function becomes

$$h = \underline{h}_0 (T_c^2/T_0^2 - 1) \chi^{-3}, \tag{62}$$

and that means

$$h = \frac{1/3}{t} \tag{63}$$

Thus, it yields $a \approx t^{1/3}$ which corresponds to a *stiff-matter driving source* $p = w$ (i.e. with $\gamma = 2$) that gives a constant deceleration $q_{\gamma=2} \equiv 2$ exactly as it is predicted by (58.b) for small χ. Hence, in the very early moments, the terms in $\sin h^{-2}\chi \cong \chi^{-2}$ contained in the parenthesis of (60.c,d) become dominant leading to $w_s \cong p_s \cong \mathrm{const.}/a^6$ and the background radiation gets negligible, leaving ourselves with a $k = 0$ -RW Universe dominated by a coherent massless scalar field which behaves phenomenologically as a stiff-matter source (Gottlober *et al.* 1990).

In the opposite asymptotic region, for large values of χ, we get

$$\begin{aligned} &\text{a)} \qquad t \cong \frac{1}{8}\,\underline{h}_0^{-1}\,\frac{(T_0/T_c)^2}{1-(T_0/T_c)^2}\mathrm{e}^{2\chi} \\ &\text{b)} \qquad h \cong 4\underline{h}_0(T_c^2/T_0^2 - 1)\mathrm{e}^{-2\chi} \end{aligned} \tag{64}$$

i.e.

$$h \cong \frac{1/2}{t} \tag{65}$$

and naturally,

$$\begin{aligned} &\text{a)} \qquad a(t) \cong (2\underline{h}_0 t)^{1/2} \\ &\text{b)} \qquad q \cong 1, \end{aligned} \tag{66}$$

meaning that we deal with a radiation dominated Universe with

$$w = w_r + w_s \cong \frac{\pi^2}{10}T^4; \;\; p = p_r + p_s \cong \frac{\pi^2}{30}T^4 \tag{67}$$

where

$$T \cong \frac{T_0}{\sqrt{2\underline{h}_0 t}}. \tag{68}$$

Now, from a somewhat restrictive point of view, it can be noticed that, as we have started the calculus of the thermalization of the massless scalar field ϕ with a Hubble constant h_0 corresponding to a thermal background radiation, it would be natural for it to be as close as possible to $\underline{h}_0$ which means that, having a look at (51), the squared ratio of the Minkowskian temperature T_0 to the Planckian one T_c has to be much smaller than 1, i.e. an $O\left[\left(\frac{T_0}{T_c}\right)^2\right]$. That seems to be a very natural assumption as long as we did not quantize the gravitational field (for dealing with a consistent model at the Planck scale) and, moreover, we did not take into consideration, within the framework of the matrix density formalism of thermalization, the contributions due to the "off-diagonal" parts $:H_\pm:$ of the total Hamiltonian

$$:H: = :H_0: + :H_+: + :H_-:, \tag{69}$$

where

$$\text{a)} \qquad :H_0: = (2h_0 t)^{-1/2}\int \frac{\mathrm{d}^3 k}{|\boldsymbol{k}|}\left[|\boldsymbol{k}|^2 + \frac{h_0/2}{2h_0 t}\right] a_{\boldsymbol{k}}^+ a_{\boldsymbol{k}}$$

$$\text{b)} \qquad :H_0: = :H_-^+: = \frac{ih_0/2}{2h_0 t} \int \frac{\mathrm{d}^3\boldsymbol{k}}{|\boldsymbol{k}|} \left[|\boldsymbol{k}|^2 - \frac{ih_0/2}{\sqrt{2h_0 t}} \right] \mathrm{e}^{\mp 2i \frac{|\boldsymbol{k}|}{h_0} \sqrt{2h_0 t}} (a_{\boldsymbol{k}})^2 \tag{70}$$

derived from the 44-component of (29.b). Therefore, let us consider $T_0 \ll T_c$ and see how the model looks from the point of view of the so-called "predictive power". Summarizing all the important formulas, namely (56-60), adapted to $T_0 \ll T_c$ with T_c given by (53) and $\underline{h}_0$ given by (52), we get

$$\begin{aligned}
&\text{a)} \qquad a(\chi) = \frac{\sqrt{\pi}}{3} T_0 \sin\mathrm{h}\chi \\
&\text{b)} \qquad t(\chi) = \frac{\sqrt{15/\pi}}{36} \left[\frac{1}{2} \sin\mathrm{h}(2\chi) - \chi \right] \\
&\text{c)} \qquad h(\chi) = 18\sqrt{\frac{\pi}{15}} \frac{\cos\mathrm{h}\chi}{\sin\mathrm{h}^3\chi} \\
&\text{d)} \qquad q(\chi) = 2 - \tanh^2\chi \\
&\text{e)} \qquad T(\chi) = \frac{3/\sqrt{\pi}}{\sin\mathrm{h}\chi} \\
&\text{f)} \qquad w_r = \frac{27/5}{\sin\mathrm{h}^4\chi} \\
&\text{g)} \qquad p_r = \frac{9/5}{\sin\mathrm{h}^4\chi} \\
&\text{h)} \qquad w_s = \frac{27/10}{\sin\mathrm{h}^4\chi} \left(1 + \frac{3}{\sin\mathrm{h}^2\chi} \right) \\
&\text{i)} \qquad p_s = \frac{9/10}{\sin\mathrm{h}^4\chi} \left(1 + \frac{9}{\sin\mathrm{h}^2\chi} \right)
\end{aligned} \tag{71}$$

As it can be noticed from (71.c), the value of χ, denoted by χ_c, for which $T(\chi_c) = T_c$ is given by

$$\chi_c = \ln\left(\sqrt{2} + 1\right)$$

and it leads *indeed* to a *subPlanckian moment* (from (71.b))

$$t_c = \frac{1}{6}\sqrt{\frac{5/6}{\pi}} \left[1 - \frac{1}{\sqrt{2}} \ln\left(\sqrt{2} + 1\right) \right] \tag{72}$$

when the temperature reached the "critical" value T_c. So, the model is safe enough from this point of view since, staying away from such very early moments as (72), we will deal with values of χ much larger than χ_c and, consequently, the temperature will be much smaller than T_c, exactly as we have already assumed for T_0.

At a principial observation level, it is clear that the knowledge obtained by a measurement of a value T_* of the background radiation temperature will determine the χ_* parameter as

$$\chi_* = \ln\left[\frac{3/\sqrt{\pi}}{T_*} + \sqrt{\frac{9/\pi}{T_*^2} + 1}\right]$$

and consequently, the cosmic time

$$t_* = \frac{1/2}{2\pi\sqrt{\frac{\pi}{15}}T_*^2}\left[\sqrt{1+\frac{\pi}{9}T_*^2} - \frac{\pi}{9}T_*^2\ln\left(\frac{3/\sqrt{\pi}}{T_*}\left(1+\sqrt{1+\frac{\pi}{9}T_*^2}\right)\right)\right], \tag{73}$$

the Hubble constant

$$h_* = 2\pi\sqrt{\frac{\pi}{15}}T_*^2\left(1+\frac{\pi}{9}T_*^2\right)^{1/2} \tag{74}$$

and the deceleration parameter

$$q_* = 2 - \left(1+\frac{\pi}{9}T_*^2\right)^{-1} \tag{75}$$

If the measurement is done at moments much later than t_c of (72) then $T_* \ll \frac{3}{\sqrt{\pi}}$ and the above formulas simplify to

$$\begin{aligned} &\text{a)} \qquad t_* \cong \frac{1}{2}h_*^{-1} \\ &\text{b)} \qquad h_* \cong 2\pi\sqrt{\frac{\pi}{15}}T_*^2 \\ &\text{c)} \qquad q_* \cong 1 \end{aligned} \tag{76}$$

Accordingly, the corresponding expressions of the thermodynamical quantities

$$\begin{aligned} &w_r = \frac{\pi^2}{15}T_*^4; \quad p_r = \frac{\pi^2}{45}T_*^4 \\ &w_s = \frac{\pi^2}{30}T_*^4\left(1+\frac{\pi}{3}T_*^2\right); \quad p_s = \frac{\pi^2}{90}T_*^4\left(1+\pi T_*^2\right) \end{aligned} \tag{77}$$

simplify to the expected formulas

$$w_r = 3p_r = \frac{\pi^2}{15}T_*^4$$

$$w_s = 3p_s = \frac{\pi^2}{30}T_*^4$$

However, one of the most important cosmological relations, namely the luminosity-distance versus the cosmological red-shift, adequately obtained from (Weinberg 1972)

$$d_L = \frac{a(t_{0b})}{a(t)}a(t_{0b})\int^{0b}\frac{d\tau}{a(\tau)} \text{ and } z = \frac{a(t_{0b})}{a(t)} - 1,$$

explicitly reads

$$d_L(z) = \frac{h_{0b}^{-1}}{2 - q_{0b}} \left[(1 + z) - \sqrt{(q_{0b} - 1)(1 + z)^2 + (2 - q_{0b})}\,\right] \tag{78}$$

and fully characterizes the presented model independently of the approximation in $O\left[(T_0/T_c)^2\right]$.

Acknowledgements. One of the authors (C.D.) wishes to express his gratitude to the Japanese Government for financially supporting his research during an 18-months Monbusho Fellowship Programme. The authors would like to thank their colleagues from Toyama University Quantum Theory Group, where part of this work has been done and its main results have been discussed in a friendly and enlighting atmosphere. Last but not least, special thanks go to Professor Tomoaki Kawaguchi for his warm hospitality at Tsukuba International University, where the pre-final form of this paper was presented in a colloquial session.

REFERENCES

Birell N.D., Davies P.C.W.: 1982, *Quantum Fields in Curved Space*, Cambridge Univ. Press, Cambridge.

Gottlober S., Haubold H.J., Mûcket J.P., Muller V.: 1990, *Early Evolution of the Universe and Formation of Structure*. Akademie-Verlag, Berlin.

Kaku M.: 1993, *Quantum Field Theory. A Modern Introduction*, Oxford University Press, Oxford.

Leigh, R.G., Rattazzi, R.: 1995, *Preprint het-ph/9503402*.

Linde A.: 1990, *Particle Physics and Inflationary Cosmology*, Harwood Academic Publishers, Chur.

Robertson H.P., Noonan T.W.: 1968, *Relativity and Cosmology*, W.B. Saunders, Philadelphia.

Sen, D.:1986, J. Math. Phys, **27**, 472.

Weinberg S.: 1972, *Gravitation and Cosmology*, Wiley, New York.

Received on 2 December, 1996

DIRAC FIELDS IN CURVED SPACETIME
A FORMAL APPROACH

CIPRIAN DARIESCU, MARINA-AURA DARIESCU

Department of Theoretical Physics,
"Al.I. Cuza" University, 6600 Iaşi, Romania

Abstract: Working on a general curved spacetime with Levi-Civita connection, the SO(1,3)-gauge covariant Dirac equation and its dynamical first order solution are derived using a formal algebraic approach in terms of tetrads. The obtained results can be written down for different manifolds of cosmological interest.

Key words: Dirac equation, canonical quantization in curved spacetime.

1. INTRODUCTION

Although there is not a quantum theory of the gravitational field yet, we need both general relativity and quantum field theory in order to have a consistent description of the dynamics of the Universe and its matter content. Therefore, the problem of quantizing scalar and spinorial fields on a gravitational background given by Einstein's equations is of a real interest now (Birell and Davies 1982). Recent articles have been focused on discussing the quantum particle creation effects in typical cosmological models (Leigh and Rattazzi 1995 and the references therein). But the Dirac equation being conformally invariant, like the ones describing the photons and the massless neutrinos, an isotropic expansion of the Universe does not create these particles.

In the standard cosmological models the plasma filling the early Universe is populated by both bosons and fermions. Together with the Klein-Gordon equation, the formulation of the Dirac equation on curved manifolds is very well motivated by various recent cosmological data. The construction of scalar fields from spinors being trivial, we shall study in the followings the Dirac fields. So, the aim of this paper is to write down the SO(1,3)-gauge covariant Dirac equation and to derive its dynamical first order solution. Working on a general curved spacetime, the obtained results can be put in a concrete form for different particular manifolds of cosmological interest. For example, in the case of $S^3 \times \mathbf{R}$ topology, the Dirac-type equation has been derived together with the complete set of solutions (Carmeli and Malin 1985, Sen 1986). Massive fermionic fields minimally coupled to an $S^3 \times \mathbf{R}$ manifold and to a spacetime described by a Kerr–Newman metric (Dariescu, Dariescu and Gottlieb 1995) have been also peformed. Moreover, in general Lorentz-gauge-invariant tetradic formulation of an $SU(2) \times U(1)$ gauge theory of bosons and fermions on $S^3 \times \mathbf{R}$ we have obtained the complete set of Dirac–Klein–Gordon–Maxwell–Yang–Mills equations (Dariescu and Dariescu 1994). It is worthwhile then to analyse in a forthcoming paper the general results derived in the followings for these cosmological models.

2. SETTING UP THE CONVENTIONS

The usual spacetime metric

$$ds^2 = g_{ik} dx^i dx^k \tag{1}$$

can be cast into the form

$$ds^2 = \eta_{ab} \omega^a \omega^b \tag{2}$$

by a suitable definition of the pseudo-orthonormal dual tetrad

$$\omega^a = \omega^a_i dx^i \tag{3}$$

and, correspondingly, through

$$\langle \omega^a, e_b \rangle = \delta^a_b \tag{4}$$

of the pseudo-orthonormal tetrad

$$e_a = e^i_a \frac{\partial}{\partial x^i} \tag{5}$$

Generally, the 1-form connection coefficients, Γ^c_{ab}, defined as

$$\nabla_b e_a = \Gamma^c_{ab} e_c \tag{6}$$

can be computed employing the Cartan's first structure equations

$$d\omega^a = -\Gamma^a_b \wedge \omega^b \tag{7}$$

in the absence of torsion, i.e. for a symmetric connection.

Because with respect to the rigid tensor basis $\left\{\omega^a \otimes \omega^b\right\}_{a,b=\overline{1,4}}$ the metric

$$g = \eta \tag{8}$$

is constant (Minkowskian), the metric condition of the linear connection Γ reads

$$g_{ab;c} = 0$$

i.e.

$$\Gamma_{(ab)c} = 0 \tag{9}$$

expressing the well-known skew-symmetry of connection coefficients with respect to a rigid frame in the first two indices.

According to Einstein's Equivalence Principle in Local Formulation, any Lorentz-covariant field equation in the Minkowskian spacetime can be Lorentz-gauge-covariantly generalized in curved spacetime, with respect to a pseudo-orthonormal basis, by simply replacing the partial derivatives with the SO(3,1)-gauge covariant ones:

$$\partial_a \rightarrow \nabla_a \tag{10}$$

or, in a more condensed notation which will be used in the following sections

$$(\bullet)_{,a} \rightarrow (\bullet)_{;a} \tag{11}$$

Thus, for the Minkowskian Dirac field equations

$$\gamma^i \psi_{,i} + m\psi = 0; \ \overline{\psi}_{,i}\gamma^i - m\overline{\psi} = 0 \tag{12}$$

one gets immediately the SO(3,1)-gauge covariant expression of the Dirac field equation as being

$$\gamma^a \psi_{;a} + m\psi = 0; \ \overline{\psi}_{;a}\gamma^a - m\overline{\psi} = 0 \tag{13}$$

where, as we have stressed above,

$$\psi_{;a} = \psi_{|a} + \frac{1}{4}\Gamma_{bca}\gamma^b\gamma^c\psi; \ \overline{\psi}_{;a} = \overline{\psi}_{|a} - \frac{1}{4}\overline{\psi}\gamma^b\gamma^c\Gamma_{bca} \tag{14}$$

The Dirac matrices γ, satisfying the algebra

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab} \tag{15}$$

will be considered all over this paper in the most common representation

$$\begin{aligned} &\gamma^\mu = \gamma_\mu = -i\beta\alpha^\mu \\ &\qquad\qquad\qquad , \ \mu = 1,2,3 \\ &\gamma^4 = -\gamma_4 = -i\beta \end{aligned} \tag{16}$$

with

$$\alpha^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \sigma^{\mu} & 0 \end{pmatrix} \quad ; \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{17}$$

σ^{μ} being the usual Pauli matrices. Correspondingly, $\gamma^5 = \gamma_5$ is generally defined as

$$\gamma^5 = -\frac{i}{4!}\,\varepsilon_{abcd}\,\gamma^a\gamma^b\gamma^c\gamma^d, \quad \text{with } \varepsilon_{1234} = -\varepsilon^{1234} = -1 \tag{18}$$

and it concretely reads

$$\gamma^5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{19}$$

3. DYNAMICAL FIRST ORDER SOLUTION

The aim of this section is to algebraically derive the solution of (13) in the first order approximation and to put its image in the momentum space in a convenient matrix formulation which turns to be useful in Section 4.

Therefore, we proceed by writing down the dynamical 0-order solution. Neglecting Γ in the expressions of the covariant derivatives (14), equation (13) reads

$$\gamma^a \psi_{|a} + m\psi = 0 \tag{20}$$

and it can be treated formally, at an algebraic level, as its Minkowskian equivalent

$$D_+^{(0)}\psi = 0 \quad \text{with} \quad D_+^{(0)}(x) = \gamma^i\partial_i + m \tag{21}$$

Thus,

$$D_+^{(0)}(x)[e^{ipx}] = i(\widehat{p} - im)e^{ipx} \quad \text{with} \quad \widehat{p} = \gamma \cdot p \tag{22}$$

and assuming that $\left[D_+^{(0)}(x)\right]^{-1}$ is the inverse of $D_+^{(0)}(x)$, it results

$$i\left[D_+^{(0)}(x)\right]^{-1}(e^{ipx})(\widehat{p} - im) = e^{ipx}$$

i.e.

$$\left[D_+^{(0)}(x)\right]^{-1}(e^{ipx}) = -i\,\frac{\widehat{p} + im}{p^2 + m^2}\,e^{ipx} \tag{23}$$

since

$$(\widehat{p} + im)(\widehat{p} - im) = p^2 + m^2 \tag{24}$$

Now, writing (13) as

$$\gamma^a\psi_{|a} + m\psi = -\frac{1}{4}\,\Gamma_{bca}\gamma^a\gamma^b\gamma^c\psi \tag{25}$$

i.e.

$$D_{+}(x)\psi(x) = \frac{1}{2}\,\Gamma^{*}_{a}(x)\gamma^{a}\psi(x), \tag{26}$$

where we have used the identity

$$\gamma^{a}\gamma^{b}\gamma^{c} = \eta^{ab}\gamma^{c} + \eta^{ca}\gamma^{b} + i\varepsilon^{abcd}\gamma_{d}\gamma_{5} \tag{27}$$

that casts the rhs of (25) into

$$-\frac{1}{4}\,\Gamma_{bca}\gamma^{a}\gamma^{b}\gamma^{c}\psi = \frac{1}{2}\left[\eta^{bc}\Gamma_{abc}\gamma^{a} - \frac{i}{2}\,\varepsilon^{bcd}_{a}\Gamma_{bcd}\gamma^{5}\gamma^{a}\right]\psi \tag{28}$$

and respectively defines the self-dual-like connection

$$\Gamma^{*}_{a} = \eta^{bc}\Gamma_{abc} - \frac{i}{2}\varepsilon^{bcd}_{a}\Gamma_{bcd}\,\gamma^{5} = \Gamma_{a\cdot\cdot} - \widetilde{\Gamma}_{a}\gamma^{5}, \tag{29}$$

we perform the Fourier transposition of $\Gamma^{*}(x)$ and $\psi(x)$ as

$$D_{+}(x)\psi(x) = \frac{1}{2}\frac{1}{(2\pi)^{2}}\int \Gamma^{*}_{a}(q)\gamma^{a}(\widehat{p} + im)\delta(p^{2} + m^{2})\mathrm{e}^{i(p+q)x}a(p)\,\mathrm{d}^{4}q\,\mathrm{d}^{4}p \tag{30}$$

So, in the 1-st order approximation, the solution of (13) reads

$$\psi(x) = \varphi(x) + \chi(x) \tag{31}$$

with $\varphi(x)$ as the 0-order solution of the $D_{+}(x)$ operator

$$\varphi(x) = \frac{1}{(2\pi)^{2}}\int (\widehat{p} + im)\frac{a(\vec{p})}{\sqrt{2E_{p}}}\,\mathrm{e}^{ipx}\mathrm{d}^{3}p \tag{32}$$

and

$$\chi(x) = \frac{-i/2}{(2\pi)^{2}}\int\frac{(\widehat{q}+\widehat{p}) + im}{(q+p)^{2}+m^{2}}\,\Gamma^{*}_{a}(q)\gamma^{a}(\widehat{p}+im)\delta(p^{2}+m^{2})\,\mathrm{e}^{i(p+q)x}a(p)\,\mathrm{d}^{4}q\,\mathrm{d}^{4}p \tag{33}$$

This image of $\chi(x)$ in the momentum space

$$\chi(P) = -\frac{i/2}{(2\pi)^{6}}\int\frac{(\widehat{q}+\widehat{p})+im}{(q+p)^{2}+m^{2}}\Gamma^{*}_{a}(q)\gamma^{a}(\widehat{p}+im)\delta(p^{2}+m^{2})\mathrm{e}^{i(q+p-P)x}a(p)\,\mathrm{d}^{4}q\,\mathrm{d}^{4}p\,\mathrm{d}^{4}x =$$

$$= -\frac{i/2}{(2\pi)^{2}}\int\frac{\widehat{P} + im}{P^{2} + m^{2}}\,\Gamma^{*}_{a}(P - p)\,\gamma^{a}(\widehat{p} + im)\,\delta(p^{2} + m^{2})\,a(p)\,\mathrm{d}^{4}p$$

becomes

$$\chi_{+}(P) = -\frac{i/2}{(2\pi)^{2}}\int\frac{\widehat{P} + im}{P^{2} + m^{2}}\,\Gamma^{*}_{a}(P - p)\,\gamma^{a}(\widehat{p} + im)\frac{a(p)}{\sqrt{2E_{p}}}\,\mathrm{d}^{3}p \tag{34}$$

which, for later convenience, can be expressed in the matrix formulation, i.e. as a (4×4) matrix,

$$\chi_+(P) = -\frac{i/2}{(2\pi)^2}\frac{\widehat{P}+im}{P^2+m^2}\int \Gamma^*_a(P-p)\,\gamma^a(\widehat{p}+im)\frac{1}{\sqrt{2E_p}}\,\mathrm{d}^3p \tag{35}$$

and brings (31) to

$$\psi_+(x) = \frac{\widehat{P}+im}{\sqrt{2E_p}} - \frac{i/2}{(2\pi)^2}\frac{\widehat{P}+im}{P^2+m^2}\int \Gamma^*_a(P-p)\,\gamma^a(\widehat{p}+im)\frac{1}{\sqrt{2E_p}}\,\mathrm{d}^3p \tag{36}$$

One is able now to follow the same procedure for the Dirac adjointed part in (13). Thus, starting with

$$D_-\overline{\psi}(x) = \frac{1}{2}\,\overline{\psi}(x)\,\gamma^a\Gamma^*_a(x), \tag{37}$$

where

$$D_-\overline{\psi}(x) = \overline{\psi}_{;a}\,\gamma^a - m\overline{\psi} \tag{38}$$

we write down (37) as

$$D_-\overline{\psi}(x) = \frac{1/2}{(2\pi)^4}\int \overline{a}(p)\delta(p^2+m^2)(\widehat{p}+im)\gamma^a\Gamma^*_a(q)\mathrm{e}^{-i(p-q)x}\,\mathrm{d}^4p\,\mathrm{d}^4q \tag{39}$$

Noticing that

$$D_-^{-1}(x)\left[\mathrm{e}^{-i(p-q)x}\right] = i\frac{(\widehat{p}-\widehat{q})+im}{(p-q)^2+m^2}\,\mathrm{e}^{-i(p-q)x} \tag{40}$$

we obtain

$$\overline{\chi}(x) = \frac{1/2}{(2\pi)^4}\int\overline{a}(p)\delta(p^2+m^2)(\widehat{p}+im)\gamma^a\Gamma^*_a(q)\frac{\widehat{p}-\widehat{q}+im}{(p-q)^2+m^2}\,\mathrm{e}^{-i(p-q)x}\mathrm{d}^4p\,\mathrm{d}^4q \tag{41}$$

which, in the momentum representation defined as

$$\overline{\chi}(P) = \frac{1}{(2\pi)^2}\int \overline{\chi}(x)\,\mathrm{e}^{ipx}\mathrm{d}^4x, \tag{42}$$

concretely gets

$$\begin{aligned}\overline{\chi}(P) &= \frac{1/2}{(2\pi)^4}\int\overline{a}(p)\delta(p^2+m^2)(\widehat{p}+im)\gamma^a\Gamma^*_a(q)\frac{\widehat{p}-\widehat{q}+im}{(p-q)^2+m^2}\delta(P-(p-q))\mathrm{d}^4p\,\mathrm{d}^4q = \\ &= \frac{i/2}{(2\pi)^2}\int \overline{a}(p)\delta(p^2+m^2)(\widehat{p}+im)\gamma^a\Gamma^*_a(P-p)\frac{\widehat{P}+im}{P^2+m^2}\,\mathrm{d}^4p\end{aligned} \tag{43}$$

i.e.

$$\overline{\chi}(P) = \frac{i/2}{(2\pi)^2}\int \frac{\mathrm{d}^3p}{\sqrt{2E_p}}\,\overline{a}(p)(\widehat{p}+im)\,\gamma^a\Gamma^*_a(P-p)\frac{\widehat{P}+im}{P^2+m^2} \tag{44}$$

As previously, it is convenient to express both (44) and the solution in the 1-st order approximation in matrix formulation, namely

$$\overline{\chi}_+(P) = \frac{i/2}{(2\pi)^2}\int \frac{\mathrm{d}^3p}{\sqrt{2E_p}}\,(\widehat{p}+im)\,\gamma^a\Gamma^*_a(P-p)\frac{\widehat{P}+im}{P^2+m^2} \tag{45}$$

and correspondingly

$$\overline{\psi}_+(P) = \frac{\widehat{P}+im}{\sqrt{2E_p}} + \frac{i/2}{(2\pi)^2}\int \frac{\mathrm{d}^3p}{\sqrt{2E_p}}\,(\widehat{p}+im)\gamma^a\Gamma^*_a(P-p)\frac{\widehat{P}+im}{P^2+m^2} \tag{46}$$

As it can be seen, the formalism developed here is quite general with respect to the generalized Fourier images of the one form connection components, i.e. once one has worked out the complete orthonormal set of solutions of Dirac nonperturbed equation (20) any gravitational field can be accomodated in the theory as a perturbation. However, the technical problem lies on deriving this set. It is the aim of a forthcoming paper to clarify this.

Acknowledgements. The authors would like to thank the members of the Quantum Theory Group from Toyama University, Japan, for their kind hospitality and fruitful discussions on an early version of the present paper.

REFERENCES

Birell, N.D., Davies, P.C.W.: 1982, *Quantum Fields in Curved Space*, Cambridge Univ. Press, Cambridge.

Carmeli, M., Malin, S.: 1985, *Found. Phys.*, **15**, 1019.

Dariescu, C., Dariescu, M.A.: 1994, *Found. Phys.*, **24**, 1577.

Dariescu, M.A, Dariescu, C., Gottlieb, I.: 1995, *Found. Phys.*, **25**, 957, 1523.

Received on 19 December, 1996

THE EQUILIBRIUM OF POLYTROPES IN A POLOIDAL MAGNETIC FIELD

CRISTINA BLAGA

Astronomical Institute of the Romanian Academy
Astronomical Observatory Cluj-Napoca
Str. Cireşilor, 19, 3400, Cluj-Napoca, Romania

Abstract. In this article we study the equilibrium of polytropes in a poloidal magnetic field. We see that the magnetic field is the solution of a nonhomogeneous Euler equation for which we write the general solution. It depends on the Lane–Emden function. So we determined the exact solution for $n = 0, 1$. To study the relation between the magnetic and the polytropic index we use the first order Padé approximants for the Lane–Emden function.

Key words: stellar structure, magnetic field.

1. INTRODUCTION

The detection of the magnetic field at the stellar surface (see Babcock, 1958) raised the question of its role in the equilibrium of the star. Earlier, Chandrasekhar and Fermi (1953) found, using the virial theorem, that there is a maximum value for the magnetic field above which the equilibrium is lost. The measured values are below it, so the stars are in equilibrium. But, the problem of the distribution of the magnetic field in the star raises a quite complicated mathematical problem. An analytical solution of the problem could be found only if we make some supplementary hypothesis. We shall presume, based on the photometric observations (Borra and Landstreet, 1980), that at the stellar surface the magnetic field can be approximated by a dipole with the center in the center of the star and so the field is weak and has an axial symmetry. We also consider that the star is a complete polytrope with a given polytropic index n. In this case to obtain the distribution of the magnetic field in the star we have to solve a singular Sturm-Liouville problem (see Roxburgh, 1966). But in the particular case of the poloidal magnetic field (see the decomposition proposed by Lüst and Schlüter, 1954), the problem can be easily solved, because it yields a nonhomogeneous Euler equation. Its general solution will depend on the Lane–Emden function. We will see that the first order Padé approximants, proposed by Pascual (1977), provide us a good approximation of the problem and its analytic form allows us a qualitative interpretation.

The equilibrium of a polytrope in a magnetic poloidal field was studied by Monaghan (1965). He found the differential equation that provides the magnetic field, solved it numerically for different polytropic indices and discussed the influence of the polytropic index n on the distribution of the magnetic field. Further, we will show that Monaghan's differential equation for the poloidal magnetic field is equivalent to an Euler equation and write its general solution. The Lane–Emden function present in this general solution will be substituted by its first order Padé approximants (Pascual, 1977), fact that permits us a comparison between this solution and a numerical one.

2. BASIC EQUATIONS

To describe the equilibrium in a poloidal magnetic field we will use the equation of the hydromagnetic equilibrium, the Poisson equation, the Ampère law, the magnetic monopole equation and the polytropic relation, respectively:

$$\frac{\nabla P}{\rho} = -\nabla\phi + \frac{\boldsymbol{j}\times\boldsymbol{H}}{c\rho}, \tag{1}$$

$$\nabla^2\phi = 4\pi G\rho, \tag{2}$$

$$\mathrm{curl}\,\boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{j}, \tag{3}$$

$$\mathrm{div}\,\boldsymbol{H} = 0, \tag{4}$$

$$P = K\rho^{1+\frac{1}{n}}, \tag{5}$$

in which the notations are usual. Supposing that the star has axial symmetry and using the spherical coordinates (r, θ, ϕ) with $r = 0$ in the center of the star and $\theta = 0$, the symmetry axis, we obtain that $\frac{\partial}{\partial\phi} = 0$. Having in mind the representation of the general solution of (4) (see Chandrasekhar, 1961) and the simplifications due to the hypothesis of axial symmetry and poloidal magnetic field, we will obtain that $\boldsymbol{H} = (H_r, H_\theta, 0)$, with:

$$H_r = \frac{1}{r^2\sin\theta}\frac{\partial S}{\partial\theta} \quad ; \quad H_\theta = -\frac{1}{r\sin\theta}\frac{\partial S}{\partial r}. \tag{6}$$

Taking curl of equation (1) and using (3) we will obtain:

$$\mathrm{curl}\left(\frac{\boldsymbol{H}\times \mathrm{curl}\boldsymbol{H}}{\rho}\right) = 0, \tag{7}$$

relation that will be used further to express the dependence of the magnetic field on the radius. We shall assume, in the first approximation, that the magnetic field is weak, so it presence will not modify the mass density distribution in the star. So, $\rho = \rho_0(r)$, where ρ_0 is known from the equilibrium of an unperturbed polytrope.

We will consider that the exterior magnetic field is dipolic and in the interior it is expressed by:

$$S(r,\theta) = A(r)\sin^2(\theta). \tag{8}$$

To facilitate the evaluation of the magnetic field we introduce the following transformations:

$$r = a\xi \quad , \quad \rho_0 = \rho_c\theta_n^n \quad , \quad a = \frac{K(n+1)\rho^{\frac{1}{n}-1}}{4\pi G} \quad , \quad A = D\rho_c a^4 \gamma_n \tag{9}$$

in which we recognize the Emden variables (ξ, θ_n), introduce a new dimensionless function $\gamma_n(\xi)$ proportional to the magnetic field and use the constant of integration D (see Roxburgh, 1966). This substitution allows us to build the dimensionless form of (7). So, using (6), (8) and (9) in (7), we get the following nonhomogeneous second order differential equation:

$$\frac{\partial^2\gamma_n}{\partial\xi^2} - \frac{2\gamma_n}{\xi^2} = -\theta_n^n\xi^2 \tag{10}$$

where θ_n is the Lane–Emden function of index n, i.e. the solution of the Lane–Emden equation of order n:

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta_n}{\mathrm{d}\xi}\right) = -\theta_n^n, \tag{11}$$

The boundary conditions for the equation (10) are:

$$\gamma_n = \gamma_n' = 0 \quad \text{in} \quad \xi = 0 \quad \text{and} \quad \xi\frac{\mathrm{d}\gamma_n}{\mathrm{d}\xi} + \gamma_n = 0 \quad \text{in} \quad \xi = \xi_1, \tag{12}$$

The first condition says that the magnetic field must be finite in the center of the star and the second in $\xi = \xi_1$ (with ξ_1 the polytropic radius, i.e. the first zero of the Lane–Emden function) reflects that at the surface of the star there is no discontinuity in the magnetic field.

3. THE GENERAL SOLUTION OF EQUATION (10)

The equation (10) is a nonhomogeneous Euler equation. With the transformation $\xi = e^t$ it becomes an equation with constant coefficients:

$$\frac{d^2\widetilde{\gamma}}{dt^2} - \frac{d\widetilde{\gamma}}{dt} - 2\widetilde{\gamma} = -\widetilde{\theta}^n e^{4t}, \tag{13}$$

where $\widetilde{\gamma}(t) = \gamma_n(e^t)$, $\widetilde{\theta}(t) = \theta_n(e^t)$. The fundamental system of solutions for (13) is:

$$\left\{e^{-t} \quad , \quad e^{2t}\right\}, \tag{14}$$

so the general solution is:

$$\widetilde{\gamma}(t) = c_1(t)e^{-t} + c_2(t)e^{2t}, \tag{15}$$

where the functions $c_1(t)$ and $c_2(t)$ are determined from the following conditions:

$$\begin{cases} c_1'(t)\cdot e^{-t} + c_2'(t)\cdot e^{2t} = 0 \\ -c_1'(t)\cdot e^{-t} + 2c_2'(t)\cdot e^{2t} = -\theta_n^n(t)e^{4t} \end{cases} \tag{16}$$

Solving (16) and substituting in (15) we find that the general solution is:

$$\widetilde{\gamma}_n(t) = -\frac{1}{3}e^{2t}\int e^{2t}\widetilde{\theta}_n^n(t)dt + \frac{1}{3}e^{-t}\int e^{5t}\widetilde{\theta}_n^n(t)dt + K_1e^{2t} + K_2e^{-t} \tag{17}$$

where the constants K_1 and K_2 are determined using the boundary conditions (12).

Going back from the variable t to ξ we find the following expression for $\gamma_n(\xi)$:

$$\gamma_n(\xi) = -\frac{1}{3}\xi^2\int \xi\theta_n^n(\xi)d\xi + \frac{1}{3}\frac{1}{\xi}\int \xi^4\theta_n^n(\xi)d\xi + K_1\xi^2 + \frac{K_2}{\xi}. \tag{18}$$

which enables us to find the constants K_1 and K_2 using (12):

$$K_1 = \left[\frac{1}{3}\int \xi\theta_n^n(\xi)d\xi\right]\Bigg|_{\xi=\xi_1} ; \quad K_2 = -\left[2\int \xi^4\theta_n^n(\xi)d\xi\right]\Bigg|_{\xi=0} . \tag{19}$$

We note that in (18) and (19) is involved the Lane-Emden function of index n. But as we know, there are only three cases ($n \in \{0, 1, 5\}$) whose exact form could be written (see Chandrasekhar, 1939). Further we will use an approximate form of it to be able to compare our results with the numerical results of Monaghan (1965).

4. CERTAIN SOLUTION FOR EQUATION (10)

Let us substitute in (18) $n = 0, 1$ to get exact solutions for (10). For $n = 0$ we get the solution found by Ferraro (1954), and for $n = 1$ the Monaghan's solution (1965):

for $n = 0$

$$\theta_0 = 1 - \frac{1}{6}\xi^2\gamma_n(\xi) = \xi^2 - \frac{\xi^4}{10}$$

for $n = 1$

$$\theta_1 = \frac{\sin\xi}{\xi}\gamma_n(\xi) = \xi\sin\xi + 2\cos\xi - \frac{2\sin\xi}{\xi} + \frac{1}{3}\xi^2$$

For other values of n we know only approximate or numerical solutions for (11). In the sequel we are going to use the first order Padé approximation in $\xi = 0$ for the Lane–Emden function[1] (Pascual, 1977), which means:

$$\theta_n^{[1,1]}(\xi) = \frac{60 + (3n - 10)\xi^2}{60 + 3n\xi^2}. \tag{20}$$

Choosing $n = 1$ in (18) and (22) we find the following expression for $\gamma_1(\xi)$

$$\gamma_1(\xi) = \frac{1}{3}\xi^2\left(138.3339469 + \frac{7}{6}\xi^2 - \frac{100}{3}\ln(60 + 3\xi^2)\right) \tag{21}$$

$$+\frac{1}{3}\,\frac{-\frac{7}{15}\xi^5 + \frac{200}{9}\xi^3 - \frac{4000}{3}\xi + \frac{8000}{3}\sqrt{5}\arctan\left(\frac{1}{10}\xi\sqrt{5}\right)}{\xi} \tag{22}$$

For $n = 2$, we get $\gamma_2(\xi)$

$$\gamma_2(\xi) = \frac{1}{3}\xi^2\left(-37.98177011 - \frac{2}{9}\xi^2 + \frac{1250}{9}\frac{1}{10 + \xi^2} + \frac{100}{9}\ln(10 + \xi^2)\right) + \frac{1}{3} \tag{23}$$

$$\times\left(\frac{4}{45}\xi^5 - \frac{200}{27}\xi^3 + 500\xi + \frac{12500}{9}\frac{\xi}{10 + \xi^2} - \frac{5750}{9}\sqrt{10}\arctan\left(\frac{1}{10}\xi\sqrt{10}\right)\right)/\xi \tag{24}$$

For $n = 3$ we obtain $\gamma_3(\xi)$

$$\gamma_3(\xi) = \frac{1}{3}\xi^2\left(.7999286730 + \frac{1}{1458}\xi^2 + \frac{2000000}{2187}\frac{1}{(20 + 3\xi^2)^2} - \frac{20000}{729}\frac{1}{20 + 3\xi^2}\right. \tag{25}$$

[1] In the neighbourhood of $\xi = 0$, the power series expansion of the Lane–Emden function has only even terms. So, we have to approximate a power series on ξ^2. So, in fact, here θ_n is rather a function of ξ^2 than a function of ξ. This the reason why $\theta_n^{[1,1]}$ is termed "first order Padé approximant" it being really of first order with respect to ξ^2 (not with respect to ξ).

$$-\frac{100}{729}\ln(20+3\xi^2)\Big) + \frac{1}{3}\left(-\frac{1}{3645}\xi^5 + \frac{200}{2187}\xi^3 - \frac{44000}{2187}\xi \right. \tag{26}$$

$$+\frac{40000000}{6561}\frac{\xi}{(20+3\xi^2)^2} - \frac{6200000}{6561}\frac{\xi}{20+3\xi^2} \tag{27}$$

$$\left. +\frac{76000}{2187}\sqrt{15}\arctan\left(\frac{1}{10}\xi\sqrt{15}\right)\right)/\xi \tag{28}$$

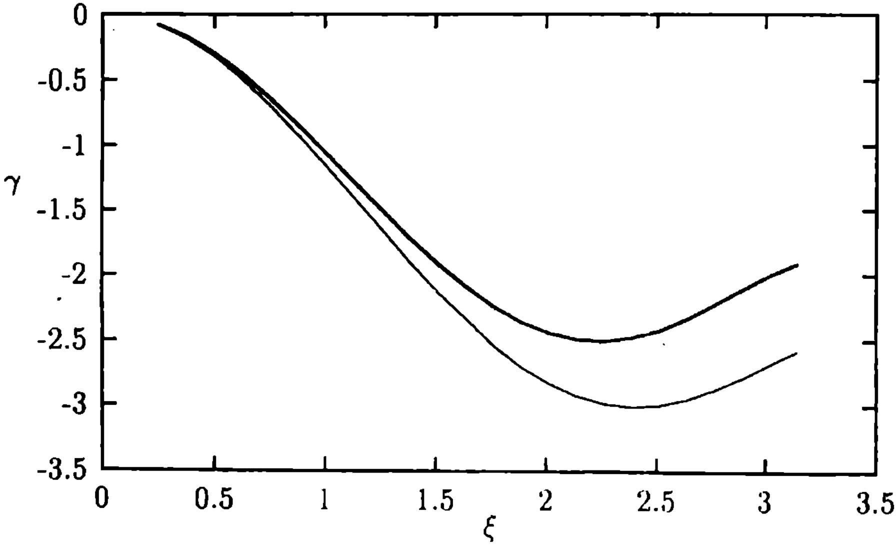

Fig. 1. — The poloidal magnetic field for $n = 1$.

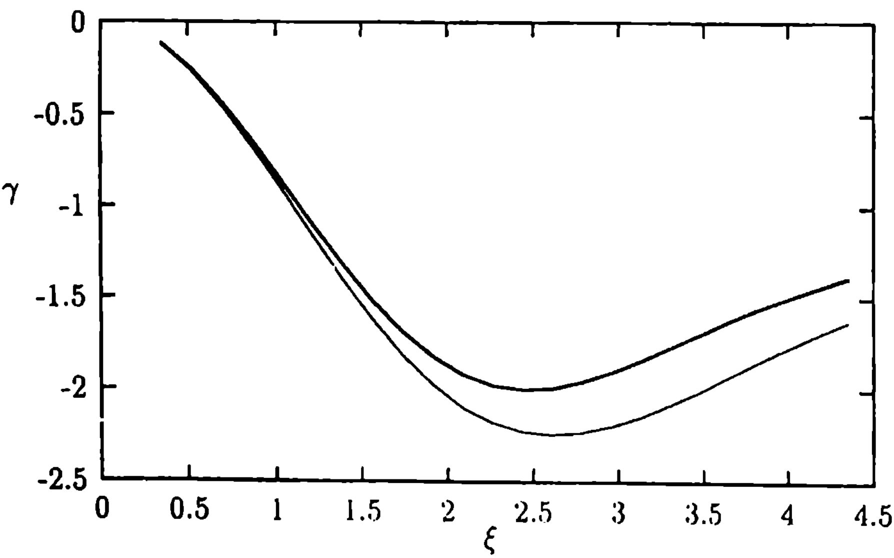

Fig. 2. — The poloidal magnetic field for $n = 2$.

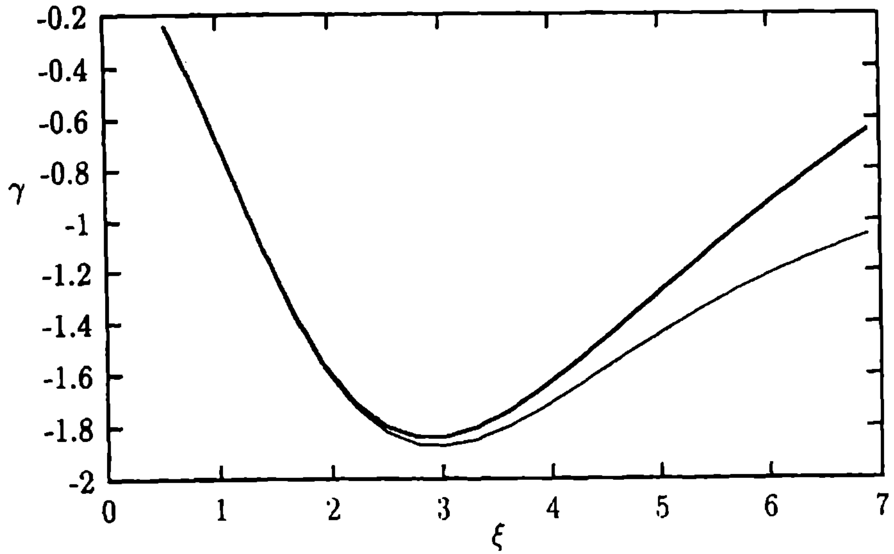

Fig. 3. — The poloidal magnetic field for $n = 3$.

In figures 1, 2 and 3 we have represented (23), (24) and (25) (thin line) and also the numerical solution (Monaghan, 1965)[2] (thick line). Our solution approximates the numerical one with the mean error 0.3, 0.17 and 0.11 respectively. At the surface of the star our solution is far from the numerical one, mainly, because we used a first order Padé approximant in $\xi = 0$ for θ_n. To overcome this fact we could use a higher order Padé approximation or to part the star in envelopes in which we compute different Padé approximants. But these approaches involved more calculations and complicated the analytical form of the solution.

In figure 4 we analyze the magnetic field at different depths in the star (on x-axis we represent ξ/ξ_1, where ξ_1 is the first zero of the Lane-Emden function) and we see that increasing the polytropic index, the main value of the field is found deeper in the star. The maximum value of the field does not depend strongly on the polytropic index for stars with strong central condensation, as we can see in figure 4.

[2] The functions $\gamma_n(\xi)$ and $B_0(\xi)$ from Monaghan's article - dimensionless functions - are in the ratio -2. So, in fact, we represent in our pictures $-2\gamma_n(\xi)$ and $B_0(\xi)$.

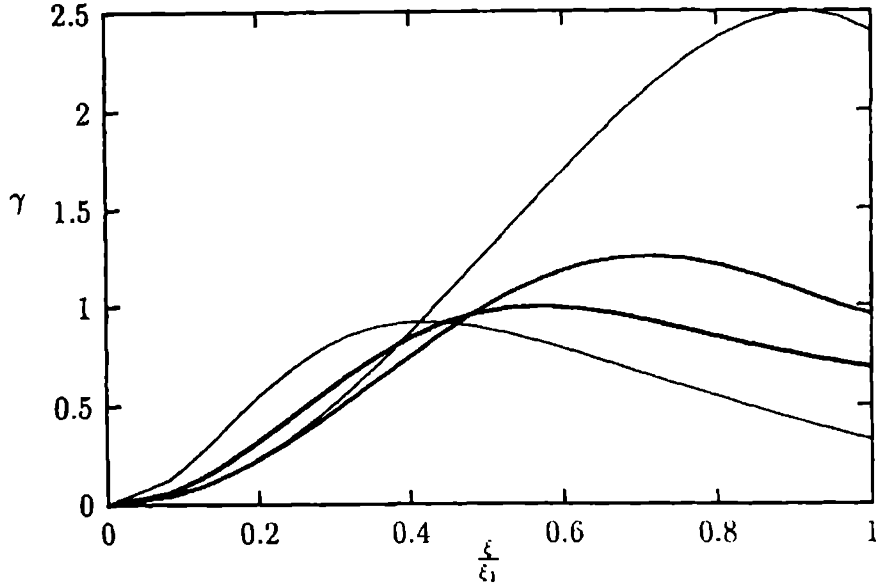

Fig. 4. — The magnetic field versus the star depth for $n \in \{0, 1, 2, 3\}$; the first line from the top is for $n = 0$.

REFERENCES

Babcock, H.W.: 1958, *Astrophys. J. Suppl.*, **3**, 141.

Borra, E.F., Landstreet, J.D.: 1980, *Astrophys. J. Suppl.*, **42**, 421.

Chandrasekhar, S.: 1939, *An Introduction to the Study of Stellar Structure*, Chicago University Press, Chicago.

Chandrasekhar, S., Fermi, E.: 1953, *Astrophys. J.*, **118**, 116.

Chandrasekhar, S.: 1961, *Hydrodynamic and Hydromagnetic Stability*, Clarendon Press, Oxford.

Ferraro, V.A.C.: 1954, *Astrophys. J.*, **119**, 407.

Lüst, R., Schlüter, A.: 1954, *Z. Astrophys.*, **34**, 263.

Monaghan, J.J.: 1965, *Mon. Not. Roy. Astron. Soc.*, **131**, 105.

Pascual: 1977, *Astron. Astrophys.*, **60**, 161.

Roxburgh, I.W.: 1966, *Mon. Not. Roy. Astron. Soc.*, **132**, 347.

Received on 19 December, 1996

NEW EXACT SOLUTIONS IN THE PLANAR, SYMMETRICAL $(n+1)$-BODY PROBLEM

EUGENIU GREBENICOV

Institute for Supercomputers of the Russian Academy of Sciences, 37, Vavilova St., Moscow 117312, Russia
e-mail: greben@ext1.ccas.ru

Abstract. One proves that, if n equal masses are initially situated at the vertices of a regular polygon centered in the $(n+1)$th mass, and if the initial velocities form a vector field symmetrical with respect to the central mass, then the configuration of regular polygon is kept all along the motion, but with variable sides and variable rotation around the centre. The motion of every mass relative to the centre is provided by the solution of the classical two-body problem.

Key words: celestial mechanics – $(n+1)$-body problem.

Elmabsout (1988) proved that in the Newtonian planar $(n+1)$-body problem (material points: $P_0, P_1, \ldots, P_n$; masses: $m_0, m_1 = m_2 = \ldots = m_n = m$; $n \geqslant 2$) there exists an exact particular solution, geometrically representable by a regular polygon of vertices $P_1, P_2, \ldots, P_n$ and centre P_0, of constant sides, rotating around P_0 with constant angular velocity. He also studied the stability of this configuration for different relations among the parameters m_0, m and n (see Elmabsout 1990, 1994, 1996).

In this paper we point out the existence of a new class of exact solutions for the Newtonian planar $(n+1)$-body problem.

THEOREM. *If the bodies P_k, $k = \overline{1,n}$, of masses $m_k = m$, $k = \overline{1,n}$, $n \geqslant 2$, form at the initial instant $t = 0$ a regular polygon centered in the body P_0 of mass m_0, and if their initial velocities form a symmetrical vector field around P_0, then for every $t \in D$ (D = maximal interval of existence for the solution; $0 \in D$) the bodies P_k will form a regular polygon centered in P_0, homothetic with the initial polygon, and rotating around P_0 with variable angular velocity. Each body P_k moves with respect to P_0 on a conic section corresponding to the solution of the classical two-body problem.*

Rom. Astron. J., Vol. 7, No. 2, p. 151–156, Bucharest, 1997

To prove the theorem, we need two lemmas.

LEMMA 1. *For $n \geqslant 2$ and $k = \overline{2,n}$, the following identities hold*

$$\begin{gathered}
\sum_{i=1, i\neq k}^{n} \cos\frac{2\pi(i-k)}{n} - \sum_{s=2}^{n} \cos\frac{2\pi(s-1)}{n} \equiv 0, \\
\sum_{i=1, i\neq k}^{n} \sin\frac{2\pi(i-k)}{n} - \sum_{s=2}^{n} \sin\frac{2\pi(s-1)}{n} \equiv 0, \\
\sum_{i=1, i\neq k}^{n} \left|\sin\frac{\pi(i-k)}{n}\right|^{-1} - \sum_{s=2}^{n} \left(\sin\frac{\pi(s-1)}{n}\right)^{-1} \equiv 0, \\
\frac{1}{2}\sum_{i=1, i\neq k}^{n} \sin\frac{2\pi(i-k)}{n}\left|\sin\frac{\pi(i-k)}{n}\right|^{-3} - \sum_{s=2}^{n} \cos\frac{\pi(s-1)}{n}\left(\sin\frac{\pi(s-1)}{n}\right)^{-2} \equiv 0,
\end{gathered} \tag{1}$$

Proof. Consider, for instance, the last identity. We shall have

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1, i\neq k}^{n} \sin\frac{2\pi(i-k)}{n}\left|\sin\frac{\pi(i-k)}{n}\right|^{-3} - \sum_{s=2}^{n} \cos\frac{\pi(s-1)}{n}\left(\sin\frac{\pi(s-1)}{n}\right)^{-2} = \\
&= \frac{1}{2}\sum_{i=1}^{k-1} \sin\frac{2\pi(i-k)}{n}\left|\sin\frac{\pi(i-k)}{n}\right|^{-3} + \frac{1}{2}\sum_{i=k+1}^{n} \sin\frac{2\pi(i-k)}{n}\left|\sin\frac{\pi(i-k)}{n}\right|^{-3} = \\
&= -\sum_{s=2}^{n-k+1} \cos\frac{\pi(s-1)}{n}\left(\sin\frac{\pi(s-1)}{n}\right)^{-2} - \sum_{s=n-k+2}^{n} \cos\frac{\pi(s-1)}{n}\left(\sin\frac{\pi(s-1)}{n}\right)^{-2} = \\
&= -\sum_{i=1}^{k-1} \cos\frac{\pi(i-k)}{n}\left(\sin\frac{\pi(i-k)}{n}\right)^{-2} + \sum_{i=k+1}^{n} \cos\frac{\pi(i-k)}{n}\left(\sin\frac{\pi(i-k)}{n}\right)^{-2} - \\
&= -\sum_{s=2}^{n-k+1} \cos\frac{\pi(s-1)}{n}\left(\sin\frac{\pi(s-1)}{n}\right)^{-2} - \sum_{s=n-k+2}^{n} \cos\frac{\pi(s-1)}{n}\left(\sin\frac{\pi(s-1)}{n}\right)^{-2} = \\
&= \sum_{j=n-k+2}^{n} \cos\frac{\pi(j-1)}{n}\left(\sin\frac{\pi(j-1)}{n}\right)^{-2} + \sum_{j=2}^{n-k+1} \cos\frac{\pi(j-1)}{n}\left(\sin\frac{\pi(j-1)}{n}\right)^{-2} - \\
&- \sum_{s=2}^{n-k+1} \cos\frac{\pi(s-1)}{n}\left(\sin\frac{\pi(s-1)}{n}\right)^{-2} - \sum_{s=n-k+2}^{n} \cos\frac{\pi(s-1)}{n}\left(\sin\frac{\pi(s-1)}{n}\right)^{-2} \equiv 0
\end{aligned}$$

The last equality was obtained by putting $j = n + i - k + 1$ in the first sum and $j = i - k + 1$ in the second sum. The other identities (1) can be proved in a wholly analogous way. □

Next we need the equations which describe the relative motion of the bodies P_k, $k=\overline{1,n}$, with respect to P_0. They have the form (e.g. Abalakin et al. 1976)

$$\begin{aligned}\ddot{\rho}_k-\rho_k\dot{\lambda}_k^2&=-\frac{G(m_0+m)}{\rho_k^2}+Gm\sum_{i=1,i\neq k}^{n}\left[\frac{\rho_i\cos(\lambda_i-\lambda_k)-\rho_k}{\Delta_{ki}^3}-\frac{\cos(\lambda_i-\lambda_k)}{\rho_i^2}\right],\\ \rho_k\ddot{\lambda}_k+2\dot{\rho}_k\dot{\lambda}_k&=Gm\sum_{i=1,i\neq k}^{n}\left(\frac{\rho_i}{\Delta_{ki}^3}-\frac{1}{\rho_i^2}\right)\sin(\lambda_i-\lambda_k),\ k=\overline{1,n},\end{aligned}\tag{2}$$

where (ρ_k,λ_k) are the polar coordinates of the body P_k in a plane originated in P_0, $G=$ Newtonian gravitational constant, and

$$\Delta_{ki}^2=\rho_i^2+\rho_k^2-2\rho_i\rho_k\cos(\lambda_i-\lambda_k),\ i\neq k, k=\overline{1,n}.$$

LEMMA 2. *The solution of the system* (2) *(of order* $4n$*) corresponding to the initial conditions*

$$\rho_k(0)=a_0,\ \dot{\rho}_k(0)=b_0,\ \lambda_k(0)=c_0+2\pi(k-1)/n, \dot{\lambda}_k(0)=d_0\tag{3}$$

represents at the same time the solution of n *identical systems of fourth order having the form*

$$\begin{aligned}\ddot{\rho}-\rho\dot{\lambda}^2&=-A_n/\rho^2,\\ \rho\ddot{\lambda}+2\dot{\rho}\dot{\lambda}&=0,\end{aligned}\tag{4}$$

with the initial conditions

$$\rho(0)=a_0,\ \dot{\rho}(0)=b_0,\ \lambda(0)=c_0,\ \dot{\lambda}(0)=d_0\tag{5}$$

where $\rho(t):=\rho_1(t)$, $\lambda(t):=\lambda_1(t)$ *and*

$$A_n=G\left[m_0+\frac{m}{4}\sum_{s=2}^{n}\left(\sin\frac{\pi(s-1)}{n}\right)^{-1}\right].\tag{6}$$

Proof. Let us introduce the new variables

$$u_k=\rho_k-\rho, v_k=\lambda_k-\lambda, k=\overline{2,n}.\tag{7}$$

In these variables, the system (2) can be written under the form of the following two subsystems:

$$\begin{aligned}\ddot{\rho}-\rho\dot{\lambda}^2&=-\frac{G(m_0+m)}{\rho^2}+Gm\sum_{s=2}^{n}\left[\frac{(\rho+u_s)\cos v_s-\rho}{\Delta_{1s}^3}-\frac{\cos v_s}{(\rho+u_s)^2}\right],\\ \rho\ddot{\lambda}+2\dot{\rho}\dot{\lambda}&=Gm\sum_{s=2}^{n}\left[\frac{\rho+u_s}{\Delta_{1s}^3}-\frac{1}{(\rho+u_s)^2}\right]\sin v_s,\end{aligned}\tag{8}$$

with the initial conditions (5), and

$$\begin{aligned}
&\ddot{u}_k - \rho(2\dot{\lambda}\dot{v}_k + \dot{v}_k^2) - u_k(\dot{\lambda} + \dot{v}_k)^2 = -G(m_0 - m)\left[\frac{1}{(\rho+u_k)^2} - \frac{1}{\rho^2}\right] + \\
&+Gm\sum_{i=1, i\neq k}^{n}\left[\frac{(\rho+u_i)\cos(v_i - v_k) - (\rho+u_k)}{\Delta_{ki}^3} - \frac{\cos(v_i - v_k)}{(\rho+u_i)^2}\right] - \\
&-Gm\sum_{s=2}^{n}\left[\frac{(\rho+u_s)\cos v_s - \rho}{\Delta_{1s}^3} - \frac{\cos v_s}{(\rho+u_s)^2}\right], \\
&\rho\ddot{v}_k + u_k(\ddot{\lambda} + \ddot{v}_k) + 2\dot{\rho}\dot{v}_k + 2\dot{u}_k(\dot{\lambda} + \dot{v}_k) = \\
&= Gm\sum_{i=1, i\neq k}^{n}\left[\frac{\rho+u_i}{\Delta_{ki}^3} - \frac{1}{(\rho+u_i)^2}\right]\sin(v_i - v_k) - \\
&-Gm\sum_{s=2}^{n}\left[\frac{\rho+u_s}{\Delta_{1s}^3} - \frac{1}{(\rho+u_s)^2}\right]\sin v_s,
\end{aligned} \tag{9}$$

with the initial conditions $u_k(0) = \dot{u}_k(0) = \dot{v}_k(0) = 0$, $v_k(0) = 2\pi(k-1)/n$, $k = \overline{2,n}$.

Observe that the orders of the subsystems (8) and (9) are 4 and $4n-4$, respectively. Also observe that

$$\Delta_{1s}^2 = \rho^2 + (\rho+u_s)^2 - 2\rho(\rho+u_s)\cos v_s,$$
$$\Delta_{ki}^2 = (\rho+u_i)^2 + (\rho+u_k)^2 - 2(\rho+u_i)(\rho+u_s)\cos(v_i - v_k),$$

with $s, k = \overline{2,n}$.

To prove the lemma, we shall show that, for any $\rho(t)$ of class C^1, the subsystem (9) admits the exact solution

$$u(t) := u_k(t) \equiv 0,\ v_k(t) = 2\pi(k-1)/n,\ k = \overline{2,n}. \tag{10}$$

This entails that the right-hand sides of (9) must be identically zero. Replacing (10) in (9), and taking into account the fact that in this case we have

$$\Delta_{1s} = 2\rho\sin[\pi(s-1)/n],$$
$$\Delta_{ki} = 2\rho|\sin[\pi(i-k)/n]|,$$

the right hand-sides of (9) vanish if

$$\sum_{i=1, i\neq k}^{n}\left[\frac{\cos\dfrac{2\pi(i-k)}{n} - 1}{8\left|\sin\dfrac{\pi(i-k)}{n}\right|^3} - \cos\frac{2\pi(i-k)}{n}\right] -$$

$$-\sum_{s=2}^{n}\left[\frac{\cos\dfrac{2\pi(s-1)}{n} - 1}{8\left(\sin\dfrac{\pi(s-1)}{n}\right)^3} - \cos\frac{2\pi(s-1)}{n}\right] \equiv 0.$$

$$\sum_{i=1,i\neq k}^{n}\left[\frac{1}{8\left|\sin\dfrac{\pi(i-k)}{n}\right|^{3}}-1\right]\sin\frac{2\pi(i-k)}{n}-$$

$$-\sum_{s=2}^{n}\left[\frac{1}{8\left(\sin\dfrac{\pi(s-1)}{n}\right)^{3}}-1\right]\sin\frac{2\pi(s-1)}{n}\equiv 0.$$

But, by virtue of Lemma 1, the above identities always hold. This means that the subsystem (9) admits the exact solution (10). Now, if we substitute (10) into the subsystem (8), and take into account the notation (6), after some trigonometric transformations we get equations (4). The lemma is proved. □

Proof of the theorem. From Lemma 2 it follows that the relative path of each body P_k, $k=\overline{1,n}$, is determined by the system (4). Therefore the polar radius ρ_k for each P_k satisfies the first equation (4), while the polar angle λ_k obeys the law $\lambda_k(t)=\lambda_1(t)+2\pi(k-1)/n$. So, the configuration of regular polygon is kept, but the polygon is rotating and side-changing.

Equation (4) are quite the motion equations of the classical two-body problem with masses $M_1=M_0$, $M_2=(m/4)\sum_{s=2}^{n}\sin[\pi(s-1)/n]^{-1}$, whose general integral is expressed by means of the quadratures

$$\pm\int\frac{\rho\mathrm{d}\rho}{\sqrt{2C_1\rho^2+2A_n\rho-a_0^4d_0^2}}=t+C_2,$$

$$\lambda_k(\rho)=a_0^2d_0\int\frac{\mathrm{d}\rho}{\rho\sqrt{2C_1\rho^2+2A_n\rho-a_0^4d_0^2}}+C_{4k},$$

where $2C_1=a_0^2d_0^2+b_0^2-2A_n/a_0$, C_2 and C_{4k} are arbitrary integration constants, $k=\overline{1,n}$. The theorem is proved. □

REMARK. The theorem remains valid in the more general case when the reciprocal attraction between the bodies P_k and P_i is proportional to $\Delta_{ki}^{-\alpha}$. The only difference consists in the fact that for $\alpha\neq 2$ every body P_k moves with respect to P_0 on other curves than conic sections.

REFERENCES

Abalakin, V.K., Aksenov, E.P., Grebenikov, E.A., Demin, V.G, Ryabov, Yu.A.: 1976, *Handbook on Celestial Mechanics and Astrodynamics*, Nauka, Moscow (Russian).

Elmabsout, B.: 1988, *Celest. Mech. Dyn. Astron.*, **41**, 131.

Elmabsout, B.: 1990, *Celest. Mech. Dyn. Astron.*, **49**, 219.

Elmabsout, B.: 1994, *Dyn. Stabil. Syst.*, **9**, 305.

Elmabsout, B.: 1996, *Rom. Astron. J.*, **6**, 61.

Received on 27 October, 1997

ON EVERHART METHOD

BAZIL PÂRV

"Babeş-Bolyai" University
Faculty of Mathematics and Computer Science,
Str. M. Kogălniceanu 1, 3400 Cluj-Napoca, Romania

Abstract. This paper deals with the Everhart numerical integration method, a well-known method in astronomical research. This method, a single-step one, is widely used for numerical integration of motion equations of celestial bodies. For an integration step, this method uses unequally-spaced substeps, defined by the roots of the so-called generating polynomial of Everhart's method. For this polynomial, this paper proposes and proves new recurrence formulae. The Maple computer algebra system was used to find and prove these formulae. Again, Maple seems to be well suited and easy to use in mathematical research.

Key words: celestial mechanics — Everhart method — computer algebra.

1. OUTLINE OF THE EVERHART METHOD

This method was proposed by Edgar Everhart (Everhart, 1974). It is used to solve numerically systems of differential equations of the following type:

$$\begin{cases} \ddot{x} = F_x(x, y, z, x_i, y_i, z_i, x_j, y_j, z_j, \ldots, t), \\ \ddot{y} = F_y(x, y, z, x_i, y_i, z_i, x_j, y_j, z_j, \ldots, t), \\ \ddot{z} = F_z(x, y, z, x_i, y_i, z_i, x_j, y_j, z_j, \ldots, t), \end{cases} \tag{1}$$

where by $\boldsymbol{F} = (F_x, F_y, F_z)$ we denote forces depending on time t, on position (expressed by the coordinates x, y, z) of the body whose path is to be integrated, as well as the positions of other bodies (identified by the indexes $i, j, \ldots$, and expressed by the coordinates $\boldsymbol{x}_i = (x_i, y_i, z_i)$, $\boldsymbol{x}_j = (x_j, y_j, z_j)$ and so on). One observes that for each body there are three second-order differential equations, belonging to the class IIS (S stands for special, since there are no velocities involved). Denoting by $\boldsymbol{x}$ the set of all coordinates from our problem (the so-called *n-body problem*), the equations (1) can be written as:

$$\ddot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}, t). \tag{2}$$

Rom. Astron. J., Vol. 7, No. 2, p. 157–168, Bucharest, 1997

The input data are: the initial moment $t_1 = 0$, the coordinates $\boldsymbol{x}_1 = \boldsymbol{x}(t_1)$, the velocities $\dot{\boldsymbol{x}}_1 = \dot{\boldsymbol{x}}(t_1)$, and the corresponding forces $\boldsymbol{F}_1 = \boldsymbol{F}(\boldsymbol{x}_1, t_1)$. Our goal is to determine the position and the velocity of the body for a moment $T > t_1$.

1.1. NOTATIONS

A time-series expansion of $\boldsymbol{F}$ about time $t = 0$ gives:

$$\ddot{\boldsymbol{x}}(t) = \ddot{\boldsymbol{x}} \equiv \boldsymbol{F} = \boldsymbol{F}_1 + A_1 \cdot t + A_2 \cdot t^2 + A_3 \cdot t^3 + \ldots + A_N \cdot t^N. \tag{3}$$

Integrating equation (3) one obtains successively:

$$\dot{\boldsymbol{x}}(t) = \dot{\boldsymbol{x}} = \dot{\boldsymbol{x}}_1 + \boldsymbol{F}_1 \cdot t + A_1 \cdot \frac{t^2}{2} + A_2 \cdot \frac{t^3}{3} + A_3 \cdot \frac{t^4}{4} + \ldots + A_N \cdot \frac{t^{N+1}}{N+1}, \tag{4}$$

$$\boldsymbol{x}(t) = \boldsymbol{x} = \boldsymbol{x}_1 + \dot{\boldsymbol{x}}_1 \cdot t + \boldsymbol{F}_1 \cdot \frac{t^2}{2} + A_1 \cdot \frac{t^3}{6} + A_2 \cdot \frac{t^4}{12} + \ldots + A_N \cdot \frac{t^{N+2}}{(N+1)\cdot(N+2)} \tag{5}$$

1.2. COEFFICIENTS

The integration scheme is as follows:

a) Choose a (unequally-spaced) subdivision of the interval $[0, T] : t_1 = 0, t_2, t_3, \ldots, T$.

b) Denote

$$\boldsymbol{x}_i \equiv \boldsymbol{x}(t_i), \;\; \boldsymbol{F}_i \equiv \boldsymbol{F}(\boldsymbol{x}_i, t_i). \tag{6}$$

c) Compute $\boldsymbol{F}_i$ $(i = 2, 3, \ldots)$.

d) Determine the coefficients $A_1, A_2, \ldots,$ from (3) such that equations (4) and (5) are accurate at time T. The truncated series of equation (3) is not a Taylor series because the coefficients A are not chosen to represent $\boldsymbol{F}$ as well as possible for all values of t; instead, these coefficients are chosen so that the expressions from (4) and (5) calculate $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ as accurately as possible at a particular time point T.

e) Write $\boldsymbol{F}$ as:

$$\boldsymbol{F}(t) \equiv \boldsymbol{F} = \boldsymbol{F}_1 + \alpha_1 \cdot t + \alpha_2 \cdot t \cdot (t - t_2) + \alpha_3 \cdot t \cdot (t - t_2) \cdot (t - t_3) + \ldots \tag{7}$$

Observe that:

$$\begin{aligned}
\boldsymbol{F}_1 &= \boldsymbol{F}(t_1), \\
\boldsymbol{F}_2 &= \boldsymbol{F}(t_2) = \boldsymbol{F}_1 + \alpha_1 \cdot t_2, \\
\boldsymbol{F}_3 &= \boldsymbol{F}(t_3) = \boldsymbol{F}_1 + \alpha_1 \cdot t_3 + \alpha_2 \cdot t_4 \cdot (t_3 - t_2), \\
\boldsymbol{F}_4 &= \boldsymbol{F}(t_4) = \boldsymbol{F}_1 + \alpha_1 \cdot t_4 + \alpha_2 \cdot t_4 \cdot (t_4 - t_2) + \alpha_3 \cdot t_4 \cdot (t_4 - t_2) \cdot (t_4 - t_3).
\end{aligned} \tag{8}$$

f) Denoting by $t_{nj} \equiv t_n - t_j$ $(n > j)$, from (8) one can compute the α coefficients:

$$\begin{aligned}
\alpha_1 &= (\boldsymbol{F}_2 - \boldsymbol{F}_1)/t_2, \\
\alpha_2 &= ((\boldsymbol{F}_3 - \boldsymbol{F}_1)/t_3 - \alpha_1)/t_{32}, \\
\alpha_3 &= (((\boldsymbol{F}_4 - \boldsymbol{F}_1)/t_4 - \alpha_1)/t_{42} - \alpha_2)/t_{43},
\end{aligned} \tag{9}$$

which are, in fact, divided differences:

$$\alpha_1 = [0, t_2],\ \alpha_2 = [0, t_2, t_3],\ \alpha_3 = [0, t_2, t_3, t_4].$$

g) The relationship between α and A coefficients is obtained by identifying the coefficients of the same power of t in (3) and (7):

$$\begin{aligned}
A_1 &= \alpha_1 + (-t_2) \cdot \alpha_2 + (t_2 t_3) \cdot \alpha_3 + \ldots \equiv c_{11} \cdot \alpha_1 + c_{21} \cdot \alpha_2 + c_{31} \cdot \alpha_3 + \ldots, \\
A_2 &= \alpha_2 + (-t_2 - t_3) \cdot \alpha_3 + \ldots \equiv c_{22} \cdot \alpha_2 + c_{32} \cdot \alpha_3 + \ldots, \\
A_3 &= \alpha_3 + \ldots \equiv c_{33} \cdot \alpha_3 + \ldots,
\end{aligned} \tag{10}$$

where the quantities c_{ij} are computed by:

$$\begin{aligned}
c_{ii} &= 1, && i \geqslant 1, \\
c_{i1} &= -t_i \cdot c_{i-1,1}, && i > 1, \\
c_{ij} &= c_{i-1,j-1} - t_i \cdot c_{i-1,j}, && 1 < j < i.
\end{aligned} \tag{11}$$

1.3. THE INTEGRATION SCHEME

In order to explain the integration scheme, we consider three substeps at times t_2, t_3, t_4, which are not uniformly spaced in the interval $[0, T]$, where the end of sequence t_4 need not coincide with T. In the case IIS, there are three predictor equations:

$$\boldsymbol{x}_2 = \boldsymbol{x}_1 + \dot{\boldsymbol{x}}_1 \cdot t_2 + \boldsymbol{F}_1 \cdot \frac{t_2^2}{2} + \left[A_1 \cdot \frac{t_2^3}{6} + A_2 \cdot \frac{t_2^4}{12} + A_3 \cdot \frac{t_2^5}{20}\right], \tag{12}$$

$$\boldsymbol{x}_3 = \boldsymbol{x}_1 + \dot{\boldsymbol{x}}_1 \cdot t_3 + \boldsymbol{F}_1 \cdot \frac{t_3^2}{2} + A_1 \cdot \frac{t_3^3}{6} + \left[A_2 \cdot \frac{t_3^4}{12} + A_3 \cdot \frac{t_3^5}{20}\right], \tag{13}$$

$$\boldsymbol{x}_4 = \boldsymbol{x}_1 + \dot{\boldsymbol{x}}_1 \cdot t_4 + \boldsymbol{F}_1 \cdot \frac{t_4^2}{2} + A_1 \cdot \frac{t_4^3}{6} + A_2 \cdot \frac{t_4^4}{12} + \left[A_3 \cdot \frac{t_4^5}{20}\right], \tag{14}$$

and two corrector ones (which compute position and velocity of the studied body at time point T):

$$\boldsymbol{x}(T) = \boldsymbol{x}_1 + \dot{\boldsymbol{x}}_1 \cdot T + \boldsymbol{F}_1 \cdot \frac{T^2}{2} + A_1 \cdot \frac{T^3}{6} + A_2 \cdot \frac{T^4}{12} + A_3 \cdot \frac{T^5}{20}, \tag{15}$$

$$\dot{x}(T) = \dot{x}_1 + F_1 \cdot T + A_1 \cdot \frac{T^2}{2} + A_2 \cdot \frac{T^3}{3} + A_3 \cdot \frac{T^4}{4} \tag{16}$$

The system (12)–(16) is implicit because the terms contained in square brackets are not known the first time they are needed.

1.4. THE EVERHART ALGORITHM

We present shortly the outline of Everhart algorithm for the time interval $[0, T]$. The initial conditions (position and velocity) are known at time $t_1 = 0$, and the algorithm corresponding to seventh order of integration accuracy (i.e. there are three substeps t_2, t_3, t_4 in $(0, T)$) is as follows:

a) Determine an approximate value of α coefficients;

Determine the A coefficients from (10).

b) Determine x_2 from (12).

Determine F_2 from (6).

Compute the new value of α_1 from the first equation (9).

Compute the new value of A_1 from the first equation (10).

c) Determine x_3 from (13).

Determine F_3 from (6).

Compute the new value of α_2 from the second equation (9).

Compute the new values of A_1 and A_2 from the first two equations (10).

d) Determine x_4 from (14).

Determine F_4 from (6).

Compute the new value of α_3 from the last equation (9).

Compute the new values of A_1, A_2 and A_3 from the first three equations (10).

e) Repeat steps b) thru d).

f) Determine $x(T)$ from (15).

Determine $\dot{x}(T)$ from (16).

1.5. FINDING SUBSTEPS

Everhart proved that the substeps used in the predictor equations correspond to the *Gauss-Radau* spacings (for odd orders of integration accuracy), and *Gauss-Lobatto* spacings (for even orders). These spacings are in fact the roots of a polynomial of rank n (n being the order of integration accuracy), which we call *the generating polynomial of the Everhart method*, or, shortly, *the Everhart polynomial.*

2. EVERHART POLYNOMIAL

2.1. LEGENDRE POLYNOMIALS

The Legendre polynomials, orthogonal on the interval $[-1, 1]$, are defined by the following recurrence equations:

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ P_n(x) &= \frac{2n-1}{n} \cdot x \cdot P_{n-1}(x) - \frac{n-1}{n} \cdot P_{n-2}(x), \quad n \geqslant 2. \end{aligned} \tag{17}$$

By making the variable change $x \to 2 \cdot x - 1$, one obtains the same equations for the interval $[0, 1]$:

$$\begin{aligned} L_0(x) &= 1, \\ L_1(x) &= 2 \cdot x - 1, \\ L_n(x) &= \frac{2n-1}{n} \cdot (2x-1) \cdot P_{n-1}(x) - \frac{n-1}{n} \cdot P_{n-2}(x), \quad n \geqslant 2. \end{aligned} \tag{18}$$

The definition of the Everhart polynomial is [Everhart, 1974]:

$$\begin{aligned} E_{2n}(x) &= (x-1) \cdot L_n'(x), \\ E_{2n+1}(x) &= \frac{L_n(x) + L_{n+1}(x)}{x}, \quad n \geqslant 0. \end{aligned} \tag{19}$$

The rank (subscript) of this polynomial is, as we already stated, the order of the integration method, and not its degree. One observes that there are different definitions for even and odd methods; the polynomial of even rank is defined by the derivative of the Legendre polynomial of the degree half its rank. Also, for an integer n (the number of the substeps considered), both E_{2n} and E_{2n+1} have the same degree, i.e. n.

2.2. RECURRENCE EQUATIONS

We prove that the Everhart polynomials can be determined by using the following recurrence equations:

$$\begin{aligned} &E_0(x) = 0, \\ &E_1(x) = 2, \\ &E_{2n}(x) = n \cdot (x-1) \cdot E_{2n-1}(x) + E_{2n-2}(x), \\ &E_{2n+1}(x) = \frac{2(2n+1)}{n(n+1)} \cdot E_{2n}(x) + E_{2n-1}(x), \quad n \geqslant 1. \end{aligned} \tag{20}$$

2.2.1. *Auxiliary results*

In the following we present some auxiliary results needed to prove the equations (20).

By writing (18) for degree $n+1$, one obtains the Legendre polynomial of degree $n-1$ as function of the Legendre polynomials of degree n and $n+1$:

$$L_{n-1}(x) = \frac{2n+1}{n} \cdot (2x-1) \cdot L_n(x) - \frac{n+1}{n} \cdot L_{n+1}(x), \quad n > 1 \tag{21}$$

By using this equation, it follows that:

$$n \cdot [(2x-1) \cdot L_n(x) - L_{n-1}(x)] = (n+1) \cdot [L_{n+1}(x) - (2x-1) \cdot L_n(x)]. \tag{22}$$

Multiplying (21) by $\frac{n^2}{n+1} \cdot (2x-1)$ one obtains:

$$\frac{n^2}{n+1} \cdot (2x-1) \cdot L_{n-1}(x) = \frac{n(2n+1)}{n+1} \cdot (2x-1)^2 \cdot L_n(x) - n \cdot (2x-1) \cdot L_{n+1}(x). \tag{23}$$

The following equation defines the derivative of the Legendre polynomial of degree n as function of Legendre polynomials of degree n and $n-1$:

$$\frac{1}{n} \cdot L_n'(x) = \frac{(2x-1) \cdot L_n(x) - L_{n-1}(x)}{2x \cdot (x-1)}, \quad n \geqslant 1. \tag{24}$$

One can prove this last equation by complete induction w.r.t. n (by using (22)), or directly, taking into account that (see Abramowicz and Stegun, 1964):

$$(x^2-1) \cdot P_n'(x) = n \cdot x \cdot P_n(x) - n \cdot P_{n-1}(x),$$

and making the variable change $x \to 2 \cdot x - 1$.

Now, by using (24) one can prove that the difference between two derivatives of the Legendre polynomials of consecutive degrees can be expressed as a sum of two Legendre polynomials of consecutive degrees:

$$L_n'(x) - L_{n-1}'(x) = \frac{n}{x} \cdot [L_n(x) + L_{n+1}(x)]. \tag{25}$$

The last equation we need expresses the difference of two Legendre polynomials of degrees $n+1$ and $n-1$ as function of the derivative of the Legendre polynomial of degree n; this is also obtained from (24):

$$L_{n+1}(x) - L_{n-1}(x) = \frac{2(2n+1)}{n(n+1)} \cdot x(x-1) \cdot L_n'(x). \tag{26}$$

2.2.2. *Proving recurrence equations*

By using the equations presented above, and the definition (19) of the Everhart polynomial, proving the recurrence equations (20) is quite straightforward.

For the first equation (20) we obtain:

$$E_{2n-1}(x) = \frac{1}{x} \cdot [L_{n-1}(x) + L_n(x)],$$

$$E_{2n-1}(x) = \frac{1}{n} \cdot [L_n'(x) - L_{n-1}'(x)],$$

$$E_{2n-1}(x) = \frac{1}{n} \cdot \frac{1}{x-1} \cdot [E_{2n}(x) - E_{2n-2}(x)],$$

$$E_{2n}(x) = n \cdot (x-1) \cdot E_{2n-1}(x) + E_{2n-2}(x),$$

where we used successively the second equation (19), equation (25), and the first equation (19).

The second equation (20) is obtained as follows:

$$E_{2n+1}(x) - E_{2n-1}(x) = \frac{L_n(x) + L_{n-1}(x) - L_{n-1}(x) - L_n(x)}{x} =$$

$$= \frac{L_{n+1}(x) - L_{n-1}(x)}{x},$$

$$E_{2n+1}(x) - E_{2n-1}(x) = \frac{1}{x} \cdot \frac{2n+1}{n+1} \cdot \frac{2x(x-1)}{n} \cdot L_n'(x),$$

$$E_{2n+1}(x) - E_{2n-1}(x) = \frac{2(2n+1)}{n(n+1)} \cdot E_{2n}(x),$$

$$E_{2n+1}(x) = \frac{2(2n+1)}{n(n+1)} \cdot E_{2n}(x) + E_{2n-1}(x).$$

by using the second equation (19), equation (26), and, finally, the first equation (19).

3. COMPUTER ALGEBRA

Computer algebra programs are today an important tool for mathematical research. As a matter of fact, we present the easy way of implementing some of the formulae introduced in this paper in Maple.

3.1. IMPLEMENTING LEGENDRE AND EVERHART POLYNOMIALS

In the following we present some Maple procedures which implement the equations (18), (19), and (20). Source code is commented.

3.1.1. *Legendre polynomials on* $[-1, 1]$

```
#
# LP(n,x) - Legendre polynomial of degree n in the variable x on
#           [-1, 1] computed by eqns (17)
#
LP := proc(n,x)
  local rez;
  option remember;   # store previous results
  if n = 0
    then rez := 1;
    else if n = 1
       then rez := x;
       else rez := (2*n - 1)/n * x * LP(n - 1, x) - (n - 1)/n *
                    LP(n-2, x);
    fi;
  fi;
  expand(rez)
end:
```

3.1.2. *Legendre polynomial on* [0, 1]

```
#
# LP01(n, x) - Legendre polynomial of degree n in the variable x on
#              [0, 1] computed by using eqns (17) and then by making
#              the variable change x -> 2 * x - 1
#
LP01 := proc(n, x)
  local rez, h;
  option remember;
  h := 'h';
  rez := LP(n, h);  # Legendre polynomial on [-1, 1] in the variable
                      h
  rez := expand(subs(h = 2 * x - 1, rez));   # variable change
  sort(rez)
end:
#
# L01(n, x) - Legendre polynomial of degree n in the variable x on
#             [0, 1] computed by using eqns (18)
#     L01(0, x) = 1
#     L01(1, x) = 2 * x - 1
#     L01(n, x) = (2*n-1)/n * (2*x-1) * L01(n-1, x) - (n-1)/n*L01(n-
#                 2,x)
#
L01 := proc(n, x)
  local rez;
  option remember;
  if n = 0
    then rez := 1;
    else if n = 1
       then rez := 2 * x - 1;
       else rez := (2*n-1)/n*(2*x-1)*L01(n-1,x)-(n-1)/n * L01(n-2,
                   x);
    fi;
  fi;
  sort(expand(rez))
end: # L01
```

3.1.3. *Derivative of the Legendre polynomial on* [0, 1]

```
#
# DL01(n, x) - the derivative of the Legendre polynomial of degree n
#               in x on [0, 1] computed by using equations
#     DL01(0, x) = 0
#     DL01(1, x) = 2
#     DL01(n, x) = (2 * (2 * n - 1))/ n * L01(n - 1, x) +
#                     (2 * n - 1)/ n * (2 * x - 1) * DL01(n-1, x)
#                   - (n - 1) / n * DL01(n - 2, x)
#
DL01 := proc(n, x)
  local rez;
  option remember;
  if n = 0
    then rez := 0;
    else if n = 1
        then rez := 2;
        else rez := (2 * (2 * n - 1))/ n * L01(n - 1, x) +
                    (2 * n - 1)/ n * (2 * x - 1) * DL01(n-1, x)
                  - (n - 1) / n * DL01(n - 2, x);
    fi;
  fi;
  rez := expand(rez);
  sort(rez)
end: # DL01
#
# DL01d(n, x) - the derivative of the Legendre polynomial of degree
#                n in x on [0, 1] computed using the recurrence
#                between the polynomial and its derivative
#     DL01d(0, x) = 0
#     DL01d(1, x) = 2 * L01(0, x) = 2
#     DL01d(2, x) = 6 * L01(1, x)
#     DL01d(n, x) = 2 * (2 * n - 1) * L01(n - 1, x) +
#                      DL01d(n - 2, x)
```

```
DL01d := proc(n, x)
  local rez;
  option remember;
  if n = 0
    then rez := 0;
    else if n = 1
       then rez := 2;
       else rez := 2 * (2 * n - 1) * L01(n - 1, x) + DL01d(n
                  - 2, x);
    fi;
  fi;
  rez := expand(rez);
  sort(rez)
end:  # DL01d
```

3.1.4. *Everhart polynomial*

```
#
# EP(n, x) = Everhart polynomial of rank k in the variable x
#             computed by using eqns (20)
#
EP := proc(n, x)
  local rez, k, r;
  option remember;
  if n = 0
    then rez := 0;
    else if n = 1
       then rez := 2;
       else if n = 2
          then rez := 2 * x - 2;
          else
             k := iquo(n, 2, 'r'); # k is the degree
             if r = 1                # n is odd
               then rez := (2 * (2*k + 1)) / (k * (k+1)) *
                           EP(n-1, x) + EP(n-2, x);
               else rez := k * (x - 1) * EP(n-1, x) +
                           EP(n-2, x);
             fi;
```

```
        fi;
      fi;
  fi;
  sort(expand(rez));
end:
```

3.2. IMPLEMENTING EVERHART METHOD

As we already stated, Everhart algorithm depends on the order of the integration accuracy and of the considered spacing. In all cases, the first force evaluation is at the fixed point $t = 0$; in the case of odd orders (Gauss–Radau spacings) the last evaluation is at the optimum position, while in the even ones (Gauss–Lobatto spacings) this evaluation is at $t = T$.

One can imagine a general Everhart procedure, which takes as parameter the order of integration accuracy. By using equations (20), one can generate the substeps (the roots of the Everhart polynomial) corresponding to the desired order. The next step is to apply the algorithm described in 1.4 for each integration step. This general procedure can be implemented in Maple (for testing purposes) or in a high-level programming language (like FORTRAN, C, or Pascal).

REFERENCES

Abramowitz, M. and Stegun, L.A.: 1964, *Handbook of Mathematical Functions*, US National Bureau of Standards, Appl. Math. Series **55**, Washington DC.

Everhart, E.: 1974, *Celest. Mech.*, **10**, 35.

Received on 14 July, 1997

AN EXTENSION OF APPLICABILITY DOMAIN FOR THE LC AND KS TRANSFORMATIONS (II)

DAN ŞELARU

Institute for Gravitational and Space Sciences,
Laboratory for Gravitational, Str. Mendeleev 21-25,
70168 Bucharest, Romania

Abstract. The extension of applicability domain of Levi–Civita and Kustaanheimo–Stiefel transformations obtained in the previous paper is applied in the Hill case of the three-body problem.

Key words: celestial mechanics, three-body problem (Hill case), Levi–Civita and Kustaanheimo–Stiefel transformations, action-angle variables.

1. INTRODUCTION

The extensions obtained in (*Şelaru*, 1997; hereafter Paper I) make possible the study of the general – both planar and three-dimensional – three-body problem (the last one being, as far as our knowledge goes, for the first time analysed in such a way) by means of the LC (*Levi–Civita*, 1904) transformations.

This paper tackles the general three-body problem (Hill case). For the planar, circular, restricted three-body problem, the use of the LC technique for the linearization of the equations of motion (in the unperturbed part of the Hamiltonian) is classical (*Pierce*, 1966). The LC technique is applied to two-dimensional problems, and this fact makes it apparently unusable in the general, three-dimensional problem. But, if we first resort to Jacobi coordinates, and if we use the angular momentum integral to reduce the number of degrees of freedom, this problem also becomes approachable by means of the LC transformation.

2. THREE-BODY PROBLEM (HILL CASE)

In general, the LC and KS transformations (Kustaanheimo and Stiefel, 1964) have the perturbed two-body problem as applicability domain. In the *general* three-

Rom. Astron. J., Vol. 7, No. 2, p. 169–175, Bucharest, 1997

body problem, the LC transformation is much less used (see also the comments made in (*Bhatnagar and Saha*, 1993)). The generalization obtained in Paper I offers a possibility – by combining the respective transformation with the Delaunay variables – to tackle this problem, too.

Using Jacobi coordinates, after the nodes have been reduced, the Hamiltonian of the three-body problem reads (with the notation used in (*Ferrer and Osacar*, 1994)):

$$H = \frac{1}{2M_1}(p_{11}^2 + p_{12}^2) - \frac{\mu_1 M_1}{\sqrt{q_{11}^2 + q_{12}^2}} + \tag{1}$$

$$+\frac{1}{2M_2}(p_{21}^2 + p_{22}^2) - \frac{\mu_2 M_2}{\sqrt{q_{21}^2 + q_{22}^2}} + Pert,$$

where m_0 and m_1 form a binary system, far from m_2. We have:

$$Pert = \frac{k^2(m_0 + m_1)m_2}{r_2} - \frac{k^2 m_0 m_2}{r_{02}} - \frac{k^2 m_1 m_2}{r_{12}}, \tag{2}$$

in which k stands for Gauss' gravitational constant.

The following notations will be used:

$$\begin{gathered} M_1 = m_0 m_1/(m_0 + m_1),\ M_2 = (m_0 + m_1)m_2/(m_0 + m_1 + m_2), \\ \mu_1 = k^2(m_0 + m_1),\ \mu_2 = k^2(m_0 + m_1 + m_2), \\ m_0' = m_0/(m_0 + m_1),\ m_1' = m_1/(m_0 + m_1), \\ r_1 = \sqrt{q_{11}^2 + q_{12}^2},\ r_2 = \sqrt{q_{21}^2 + q_{22}^2}, \\ C_1 = q_{11}p_{12} - q_{12}p_{11},\ C_2 = q_{21}p_{22} - q_{22}p_{21}, \end{gathered} \tag{3}$$

Since:

$$\langle r_1, r_2 \rangle = -q_{11}q_{21} + q_{12}q_{22}\frac{C^2 - C_1^2 - C_2^2}{2C_1C_2}, \tag{4}$$

(with C = the modulus of the angular momentum constant), and

$$\begin{gathered} r_{02}^2 = r_2^2 + (m_1' r_1)^2 + 2m_1'\langle r_1, r_2 \rangle, \\ r_{12}^2 = r_2^2 + (m_0' r_1)^2 - 2m_0'\langle r_1, r_2 \rangle, \end{gathered}$$

and using the Legendre polynomials PL_k, we find that (2) is equivalent to:

$$Pert = \sum_{k>2} A_k \frac{r_1^k}{r_2^{k+1}} PL_k(\cos\Theta),\ \cos\Theta = \frac{\langle r_1, r_2 \rangle}{r_1 r_2}, \tag{5}$$

the coefficients A_k depending on masses.

The motion of m_2 around the barycenter of the binary system is described by means of Delaunay coordinates (l, g, L, G); we obtain:

$$H = \frac{1}{2M_1}(p_{11}^2 + p_{12}^2) - \frac{\mu_1 M_1}{\sqrt{q_{11}^2 + q_{12}^2}} - \frac{M_2 \mu_2^2}{2L^2} + Pert. \tag{6}$$

Using classical notations in the perturbation (v = true anomaly in the two-body problem, m_2 – barycenter of (m_0, m_1), $C_2 = G$), we have:

$$\cos\Theta = -\frac{q_{11}\cos(v+g)}{\sqrt{q_{11}^2 + q_{12}^2}} + \frac{q_{12}\sin(v+g)}{\sqrt{q_{11}^2 + q_{12}^2}} \frac{C^2 - C_1^2 - G^2}{2C_1 G}. \tag{7}$$

In the extended phase space (after dropping the unnecessary subscripts), eq. (6) becomes:

$$H = \frac{1}{2M_1}(p_1^2 + p_2^2) - \frac{\mu_1 M_1}{\sqrt{q_1^2 + q_2^2}} - \frac{M_2 \mu_2^2}{2L^2} - h + Pert, \tag{8}$$

Restricting the perturbation to the first Legendre polynomial (Hill case, see (*Ferrer and Osacar*, 1994)), we get:

$$Pert = -\frac{\mu}{4\mathrm{a}^3}\left(\frac{\mathrm{a}}{r}\right)^3 \left(\left(3q_1^2 - 3q_2^2\overline{C}^2\right)\cos(2v+2g) - \right. \tag{9}$$

$$\left. -6q_1 q_2 \overline{C}\sin(2v+2g) + q_1^2 + q_2^2\left(3\overline{C}^2 - 2\right)\right), \; \mu = \frac{k^2 m_0 m_1 m_2}{(m_0 + m_1)}.$$

where $\mathrm{a} = \dfrac{L^2}{M_2^2 \mu_2}$ (do not confuse with the parameter a) is the semimajor axis of the orbit of the outer body.

REMARK 1. We did not consider necessary the introduction of the small parameter, because of the obviously perturbing character of the expression (9). The orders of the perturbation with respect to the main interaction are (for the first Legendre polynomial, the most important one):

$$\lambda_1 = \frac{m_2}{m_0 + m_1}\left(\frac{r_1}{r_2}\right)^3$$

for the two-body problem associated to the binary system, and

$$\lambda_2 = \frac{m_0 m_1}{(m_0 + m_1)^2}\left(\frac{r_1}{r_2}\right)^2$$

for the motion of the outer body with respect to the barycenter of the binary. According to the magnitudes of these parameters, different concrete physical situations can be modelled, as follows:

a) $\lambda_2 \ll \lambda_1 \ll 1$: natural satellite of a planet; for instance, in the Moon-Earth-Sun problem, $\lambda_1 = O(10^{-3})$, $\lambda_2 = O(10^{-7})$;

b) $\lambda_2 \to 0$, $\lambda_1 \ll 1$: in the limit, the inner restricted problem, or, according to the mass ratio of the binary system, a $(1+2)$-body problem;

c) $\lambda_1 \ll \lambda_2 \ll 1$: triple stellar system;

d) $\lambda_1 \to 0$, $\lambda_2 \ll 1$: planet orbiting a binary stellar system; in the limit, the outer restricted problem.

3. ACTION ANGLE VARIABLES

At this stage, the Hamiltonian has the suitable form for applying the transformations described in Paper I. The necessary conditions are obviously fulfilled. Equating the corresponding quantities, using the second set of variables obtained in (Paper I eqs. (18)), and dropping the asterisks, we get:

$$a = \frac{1}{M_1},\ b = \mu_1 M_1, s = -2hM_1,\ A = -\frac{8h}{M_1},\ B = -\frac{1}{h}, \tag{10}$$

$$L = J_3,\ G = J_4,\ l = \phi_3,\ g = \phi_4,\ F(\mathbf{J}) = -\frac{M_2\mu_2^2}{2J_3^2}.$$

$$\omega_3 = \frac{M_2\mu_2^2}{J_3^3}\frac{1}{4\left(1+\frac{1}{h}\frac{M_2\mu_2^2}{2J_3^2}\right)},$$

$$\overline{\omega_3} = \omega_3\frac{\sqrt{J^2\left(P^2-Q^2\right)^2+4P^2Q^2\left(P^2+Q^2\right)^2}}{P^2+Q^2}$$

In these variables we have:

$$C_1 = \frac{PQ\sqrt{J^2-(P^2+Q^2)^2}}{P^2+Q^2}, \tag{11}$$

and we shall denote:

$$\overline{C} = \frac{C^2-C_1^2-G^2}{2C_1G}. \tag{12}$$

The next step consists of the Fourier expansion of the perturbation, with the Hamiltonian

$$H = \sqrt{-\frac{8h}{M_1}\left(1+\frac{M_2\mu_2^2}{2hJ_3^2}\right)}J + Pert. \tag{13}$$

We shall use the classical expansions, function of Delaunay elements:

$$\left(\frac{a}{r}\right)^3\cos(2v+2g) = \sum_{j=-\infty}^{\infty} F_{1j}\cos(2g+jl),\ F_{10} = 0, \tag{14}$$

$$\left(\frac{a}{r}\right)^3 \sin(2v+2g) = \sum_{j=-\infty}^{\infty} F_{1j} \sin(2g+jl),$$

$$\left(\frac{a}{r}\right)^3 = \frac{L^3}{G^3} + \sum_{j=1}^{\infty} 2F_{2j} \cos jl;$$

Such expansions are used in Moon's motion theory and, partly, in the perturbing function expansion in the problem of artificial satellite motion (*Brouwer and Clemence*, 1961). The functions F_{ij} depend on the eccentricity of the osculating orbit of the outer body around the barycenter of the binary, hence on the Delaunay variables L and G.

REMARK 2. For sake of brevity, we shall not present here the concrete expressions of (14) as functions of the eccentricity; they can be found by using the integrals computed in (*Ahmed*, 1994) and Kepler's equation.

To obtain the Fourier expansion of the perturbation, the following relation from Paper I, eqs. (21)

$$l = \phi_3 - \overline{\omega_3} \sin(2\phi + \zeta);$$

must be replaced in (14). Using the notations (10) and the properties of the usual Bessel functions B_m, we get:

$$\left(\frac{a}{r}\right)^3 \cos(2v+2g) = \tag{15}$$

$$= \sum_{j=-\infty}^{\infty} \sum_{m=0}^{\infty} F_{1j} \left(B_{2m}(j\overline{\omega_3}) \left(\cos(2\phi_4 + j\phi_3 + 4m\phi + 2m\zeta) + \right.\right.$$

$$+ \cos(2\phi_4 + j\phi_3 - 4m\phi - 2m\zeta)) +$$

$$+ B_{2m+1}(j\overline{\omega_3}) (\cos(2\phi_4 + j\phi_3 - (4m+2)\phi - (2m+1)\zeta) -$$

$$- \cos(2\phi_4 + j\phi_3 + (4m+2)\phi + (2m+1)\zeta))), \; F_{10} = 0.$$

$$\left(\frac{a}{r}\right)^3 \sin(2v+2g) =$$

$$= \sum_{j=-\infty}^{\infty} \sum_{m=0}^{\infty} F_{1j} \left(B_{2m}(j\overline{\omega_3}) \left(\sin(2\phi_4 + j\phi_3 + 4m\phi + 2m\zeta) + \right.\right.$$

$$+ \sin(2\phi_4 + j\phi_3 - 4m\phi - 2m\zeta)) +$$

$$+ B_{2m+1}(j\overline{\omega_3}) (\sin(2\phi_4 + j\phi_3 - (4m+2)\phi - (2m+1)\zeta) -$$

$$- \sin(2\phi_4 + j\phi_3 + (4m+2)\phi + (2m+1)\zeta))), \; F_{10} = 0.$$

The second expansion (15) is similar to the first one, only the coefficients F_{1j} are replaced by F_{2j}, with $F_{20} = (J_3/J_4)^3$, and the arguments of the trigonometric functions do no longer contain the terms in ϕ_4.

The last step in this Fourier expansion consists of: substitution of q_1, q_2 with the expressions obtained by performing the usual LC transformation and finding the action-angle variables, trigonometric expansion of the products of cosines and sines, and summation over the terms having the same argument. All these expansions can be easily performed by using a symbolic processor.

For the same sake of brevity, we shall present only the doubly averaged Hamiltonian (the first step in a perturbative study) with respect to the rapid angle variables ϕ and ϕ_3:

$$\overline{H} = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \int_0^{2\pi} H \mathrm{d}\phi \mathrm{d}\phi_3;$$

$$\overline{H} = \sqrt{-\frac{8h}{M_1}\left(1+\frac{M_2\mu_2^2}{2hJ_3^2}\right)}J-$$

$$-\frac{\mu M_2^6\mu_2^3}{16J_3^3J_4^3(P^2+Q^2)^2\left(-2hM_1\left(1+\frac{M_2\mu_2^2}{2hJ_3^2}\right)\right)}\cdot \tag{16}$$

$$\cdot(-6J^2P^4+9\overline{C}^2J^2P^4+9P^8-9\overline{C}^2P^8+8J^2P^2Q^2-6\overline{C}^2J^2P^2Q^2+$$

$$+16P^6Q^2-12\overline{C}^2P^6Q^2-6J^2Q^4+9\overline{C}^2J^2Q^4+14Q^4P^4-6\overline{C}^2P^4Q^4+$$

$$+16P^2Q^6-12\overline{C}^2P^2Q^6+9Q^8-9\overline{C}^2Q^8).$$

Some specifications are needed here: obviously, due to the averaging we have performed, J and J_3 are first integrals of the motion; the averaging removes the angle ϕ_4, too, therefore J_4 is also a first integral; accordingly, the system we obtain has only one degree of freedom and is theoretically integrable. The Hamiltonian function (16), being obtained by double averaging, is somewhat equivalent to Harrington's Hamiltonian (*Ferrer and Osacar*, 1994). A detailed study of the Hamiltonian (16) exceeds the framework of this paper, and will be performed elsewhere.

We must also specify that, after applying the LC transformations, the Hamiltonian (13) is not regularized with respect to $(\mathbf{Q}, \mathbf{P})$, variables introduced through the LC transformation. For the planar three-body problem, Hill case, this regularization is performed by using the respective transformations. For the three-dimensional three-body problem tackled here, this desideratum can be reached by resorting to the KS transformations, but with two drawbacks: supplementary dimensions are introduced, and the expansion of the perturbation becomes more complicated.

4. SUMMARY AND CONCLUDING REMARKS

The paper uses the theoretical results obtained in Paper I to the general three-body problem. Many physical situations are suited to such an approach: the Moon's motion, the motion of triple stellar systems, a (1 + 2)-body problem, etc., the common condition being the presence of a close binary far from the third body. After presenting the manner in which the perturbed part of the Hamiltonian is expanded, we obtain the doubly averaged system with respect to the rapid angle variables.

The use of coordinates we propose has many advantages. The most important one consists of the possibility of an easier expansion of the perturbation (because the expansions function of orbital elements are used only for a two-body problem, and – taking into acount Remark 2 – there are no more truncations function of eccentricity). The regularization of the equations of motion in this problem can be obtained by using the KS transformations.

Acknowledgements. Thanks are due to Dr. Vasile Mioc for constant help. This research was partly supported by the Romanian Space Agency Grant 2097GR/1996.

REFERENCES

Ahmed, M.K.M.: 1994, *Astron. J.*, **107**, 1900.

Bhatnagar, K.B., Saha, L.M.: 1993, *Bull. Astr. Soc. India*, **21**, 1.

Brouwer, D., Clemence, G.M.: *Methods of Celestial Mechanics*, Academic Press, New York, London, 1961.

Ferrer, S., Osacar, C.: 1994, *Celes. Mech.*, **58**, 245.

Kustaanheimo, P., Stiefel, E.: 1964, *J. Math.*, **218**, 53.

Levi-Civita, T.: 1904, *Ann. Mat. Pura Appl.*, **9**, 1.

Pierce, D. A.: 1966, *Astron. J.*, **71**, 545.

Şelaru, D.: 1997, *Rom. Astron. J.*,**7**, 27.

Received on 7 July, 1997

BOUNDED NONCOLLISIONAL ORBITS IN GENERALIZED MANEFF-TYPE FIELDS (II)

CRISTINA STOICA

Institute for Gravitation and Space Sciences,
Gravitational Research Laboratory
21-25 Mendeleev Str., 70168, Bucharest, Romania

Abstract. In this paper we continue the study of the motion of a particle under the influence of a generalized Maneff-type potential field. Using previous results, it is established the set of orbital elements for bounded noncollisional motion.

Key words: celestial mechanics – Maneff's generalized law – orbital elements.

1. INTRODUCTION

The so-called Maneff's generalized law, which extends the gravitational model proposed by Maneff (1924, 1925, 1930 a, b), proved itself to be a rich source of investigations sharing both celestial mechanics and the theory of dynamical systems. Among such researches, with interesting results from mathematical and physical standpoints, we mention those performed by Lacomba et al. (1991), Casasayas et al. (1993), Diacu (1993, 1996), Diacu et al. (1995), Mioc and Stoica (1995 a, b, c, d), Delgado et al. (1996), Stoica and Mioc (1994, 1996 a, b, c).

Given by a potential function of the type

$$V(\mathbf{r}) = \frac{1}{r}\left(A + \frac{B}{2r}\right), \tag{1}$$

where $\mathbf{r}$ is the position vector in a frame originated in field's source, $r = |\mathbf{r}|$, whereas $A \in \mathbf{R}^*$, $B \in \mathbf{R}$, the Maneff's type model is suitable in describing various physical and astronomical situations, including Coulombian potentials, post-Newtonian fields, radiative fields generated by certain astronomical relativistic sources, and so forth.

In this paper we continue the study of the motion of a particle in a Maneff's type potential field. In a previous paper (Stoica 1995), we have already discussed the possible kinds of orbits. Also, in the bounded noncollisional case, we have performed the integration of the motion equations, deducing the extension of Kepler's third law and the equivalent of Kepler's equation. Here, preserving the framework of bounded noncollisional motion, we focus on defining a natural set of orbital elements.

2. BASIC RELATIONS

Let us consider a particle of unit mass moving under the influence of a Maneff's type potential field. Its motion is described by the initial value problem:

$$\ddot{\mathbf{r}} = -\frac{1}{r^2}\left(A + \frac{B}{r}\right)\frac{\mathbf{r}}{r};\ \mathbf{r}(t_0) = \mathbf{r}_0,\ \dot{\mathbf{r}}(t_0) = \dot{\mathbf{r}}_0. \tag{2}$$

Since the force acting on the particle is a central one, we can rewrite equation (2) in polar coordinates (r, θ), and obtain:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{1}{r^2}\left(A + \frac{B}{r}\right), \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = 0, \tag{4}$$

with initial data $(r, \theta, \dot{r}, \dot{\theta}) = (r_0, \theta_0, \dot{r}_0, \dot{\theta}_0)$. Equation (4) provides the angular momentum conservation law. Let $\mathbf{C}$ be the vectorial constant angular momentum supplied by the initial conditions, C its modulus. Also, let h be the constant of energy associated to the energy conservation law:

$$h = \frac{\dot{r}^2}{2} + \frac{C^2}{2r^2} - \frac{1}{r}\left(A + \frac{B}{2r}\right). \tag{5}$$

We already know (see Stoica 1995) that the conditions $(C^2 - B) > 0$, $A > 0$ together with $h \in [h_{cr}, 0)$, $h_{cr} := -A^2/\left[2(C^2 - B)\right]$, assure the noncollisional boundedness of the orbit. The trajectory's equation is given by:

$$r(\theta) = \frac{\dfrac{(C^2 - B)}{A}}{1 + \sqrt{1 + 2\dfrac{(C^2 - B)}{A^2}h}\cos\left[\dfrac{\sqrt{C^2 - B}}{C}(\theta - \theta_0) - \omega_i\right]}, \tag{6}$$

where the constant ω_i is introduced via equation (16) below. The boundedness condition $h \in [h_{cr}, 0)$ reverts to $0 \leqslant \sqrt{1 + 2(C^2 - B)h/A^2} < 1$, and so we are inspired to make the following notations:

$$p := \frac{C^2 - B}{A}, \tag{7}$$

$$e := \sqrt{1 + 2\frac{(C^2 - B)}{A^2}h}, \tag{8}$$

$$\alpha := \frac{\sqrt{C^2 - B}}{C},$$

in order to restrain trajectory's equation as

$$r(\theta) = \frac{p}{1 + e\cos\left[\alpha(\theta - \theta_0) - \omega_i\right]}, \tag{9}$$

with $\alpha > 0$ and $0 \leqslant e < 1$. As it can easily be seen, the function $r(\cdot)$ is bounded, having a minimum value $r_m = p/(1+e)$ and a maximum value $r_M = p/(1-e)$. The trajectory can be assimilated to the motion on a "pseudoellipse" – the trajectory on an ellipse whose pericentre is delayed if $\alpha > 1$ ($B < 0$) or hurried up if $\alpha < 1$ ($B > 0$), with the angle $2\pi(1 - 1/\alpha)$.

3. ORBITAL ELEMENTS

The further steps in defining the set of orbital elements are made, we say, in a natural way and also with the idea of recovering the classical Keplerian one for $A = \mu$, $B = 0$ (μ being the gravitational parameter in the Newtonian two-body problem).

The connection of the so-called geometric elements with the orbit's shape must be obvious. Hence, we propose as *pseudo-semimajor axis*

$$a := (r_m + r_M)/2 \tag{10}$$

and, as *pseudoeccentricity*, the quantity e defined by (8). These elements are related with the parameter given by (7) (no connection with the focal parameter of an ellipse) via $p = a(1 - e^2)$. So, we retain the formula (8) and

$$a = -\frac{A}{2h}, \tag{11}$$

easily obtainable from (7), (8) and (10).

The force field being central, the orbit will be planar. The elements i = *the inclination* and Ω = *the longitude of the ascending node* define the position of the motion plane in a reference frame $OXYZ$. In the same way as in the Keplerian motion, i and Ω are supplied by the vectorial angular momentum constant $\mathbf{C}$:

$$\cos i = \frac{C_Z}{C}, \tag{12}$$

$$\tan\Omega = \frac{C_X}{C_Y}, \tag{13}$$

where (C_X, C_Y, C_Z) is the decomposition of $\mathbf{C}$ according to $(OXYZ)$-frame.

A more delicate problem is the orientation of the orbit in its plane. Accepting as definition that the pericentre is the minimum distance of the current point to the centre of the circular annulus to which the motion is confined, we observe that after every period the direction of the pericentre is delayed (or hurried up) with the constant angle $2\pi(1 - 1/\alpha)$. So, there is no constant direction which would be able to offer the classical direction of the pericentre.

Let's suppose that at the initial moment the moving point lies at a pericentre. After a full period T (see Stoica 1995), it will be in the next pericentre whose direction is modified with respect to the initial one with angle $2\pi - 2\Theta$ ($\Theta = 2\pi/\alpha$). It is natural to consider that the pericentre is a moving one and its direction possesses the constant shifting velocity:

$$v_\omega := \frac{2\pi - 2\Theta}{T} = (2\pi - 2\Theta)\left[\frac{2\pi A}{(-2h)\sqrt{-2h}}\right]^{-1} \tag{14}$$

or

$$v_\omega = (-2h)\sqrt{-2h}\left(1 - \frac{1}{\alpha}\right)/A. \tag{15}$$

We observe that the angular velocity v_ω is zero for Keplerian motion (Newtonian field), where the direction of the pericentre is fixed. In the Maneff's type case, it is revealed that the real pericentre's direction is governed by the equation

$$\omega = \omega_i + v_\omega(t - t_0), \tag{16}$$

where ω_i is the argument of pericentre defining the pericentre's direction attached to the first considered period of the motion (via the initial conditions). It can be proposed – as the direction of the pericentre attached to each period of covering the trajectory between two real pericentres – the direction of the real pericentre at the start of a period (t_0 – the pericentre instant). This direction is considered by hypothesis to remain fixed all along the respective period until the next meeting with the real pericentre. Once with the start of the following period, the pericentre's direction changes with the angle $2\pi - 2\Theta$, remaining constant during the respective period. So, the *direction of the real pericentre at the start of a period* becomes the angle between the line of the nodes (defined by the fundamental plane OXY and the orbit's plane) and the pericentre's direction attached to the arc delimited by the two real pericentres corresponding each to its own initial radial vector.

As the fifth orbital element, which must remain constant at any time, we propose the *direction of the pericentre attached to the period of the function* $r(\cdot)$

between the initial moment and the instant of encountering the next real pericentre. We recall the observation that ω_i is not the real pericentre direction for the given trajectory, but it corresponds to the mathematical needs of defining the set of orbital elements.

It is obvious that if B vanishes, the pericentre's direction remains constant for the entire movement (v_ω becomes 0), becoming the well-known orientation element from the classical Keplerian elliptical trajectory.

The set of orbital elements is completed by ***the mean anomaly at epoch***

$$\chi_0 := nt_0 \tag{17}$$

given by the correspondent of Kepler's equation (Stoica 1995)

$$n(t - t_0) = \widetilde{E} - e \sin \widetilde{E},$$

(where $\widetilde{E}$ plays the part of the eccentric anomaly corresponding to the Kepler's standard problem), or, equivalently, by the time t_0 of the passage to pericentre.

4. CONCLUDING REMARKS

Of course, the quantitative study of the two-body problem in generalized Maneff-type fields could seem obsolete. Indeed, this problem appears (under different formulations) as an exercise in classical textbooks as those of Moulton (1923, p. 96, Problem 4) or Goldstein (1980, p. 123, Problem 14). Leaving aside the fact that the respective statement is incorrect in Goldstein's case or covers a very restricted area in Moulton's case, the results quoted in Section 1, especially the qualitative ones, show how complex the problem is in reality.

We know, on the one hand, that the analytic solution of the problem in closed form was already obtained (e.g. Delgado et al. 1996; Stoica and Mioc 1996a). On the other hand, the motion characteristics were fully described from both geometrical and physical standpoints (Diacu et al. 1997). Nevertheless, the characterization of the noncollisional bounded trajectories in Maneff-type fields by means of orbital elements introduced in this paper is – in our opinion – necessary. On their basis, as a next step, equations analogous to the Newton-Euler system can be established for the perturbed motion in such fields. Further, concrete physical and astronomical situations (as, e.g., the motion in spherical post-Newtonian fields; see Soffel 1989) can be analytically tackled starting from these equations. All these will be done elsewhere.

REFERENCES

Casasayas, J., Fontich, E., Nunes, A.: 1993, in E.A. Lacomba, J. Llibre (eds), *Hamiltonian Systems and Celestial Mechanics*, Adv. Ser. Nonlinear Mech., Vol. **4**, World Scientific, Singapore, 35.

Delgado, J., Diacu, F.N., Lacomba, E.A., Mingarelli, A., Mioc, V., Perez, E., Stoica, C.: 1996, *J. Math. Phys.*, **37**, 2748.

Diacu, F.N.: 1993, *J. Math. Phys.*, **34**, 5671.

Diacu, F.N.: 1996, *J. Diff. Eq.*, **128**, 58.

Diacu, F.N., Mingarelli, A., Mioc, V., Stoica, C.: 1995, in R.P. Agarwal (ed.), *Dynamical Systems and Applications*, World Sci. Ser. Appl. Anal., Vol. **4**, World Scientific, Singapore, 213.

Diacu, F.N., Mioc, V., Stoica, C.: 1997, *Phase-Space Structure and Regularization of Maneff-Type Problems* (submitted).

Goldstein, H.: 1980, *Classical Mechanics*, 2nd ed., Addison-Wesley, Reading, Mass.

Lacomba, E.A., Llibre, J., Nunes, A.: 1991, in T. Ratiu (ed.), *The Geometry of Hamiltonian Systems*, Proc. of a Workshop held June 5–16, 1989, Springer-Verlag, New York.

Maneff, G.: 1924, *C.R. Acad. Sci. Paris*, **178**, 2159.

Maneff, G.: 1925, *Z. Phys.*, **31**, 786.

Maneff, G.: 1930 a, *C.R. Acad. Sci. Paris*, **190**, 963.

Maneff, G.: 1930 b, *C.R. Acad. Sci. Paris*, **190**, 1374.

Mioc, V., Stoica, C.: 1995 a, *C.R. Acad. Sci. Paris*, **320**, sér.II, 467.

Mioc, V., Stoica, C.: 1995 b, *C.R. Acad. Sci. Paris*, **321**, sér.I, 961.

Mioc, V., Stoica, C.: 1995 c, *Bull. Astron. Belgrade*, No. **152**, 43.

Mioc, V., Stoica, C.: 1995 d, *Rom. Astron. J.*, **5**, 165.

Moulton, F.R.: 1923, *An Introduction to Celestial Mechanics*, Macmillan, London.

Soffel, M.: 1989, *Relativity in Astrometry, Celestial Mechanics and Geodesy*, Springer-Verlag, Berlin.

Stoica, C.: 1995, *Rom. Astron. J.*, 5, 157.

Stoica, C., Mioc, V.: 1994, *Studia Univ. Babeş-Bolyai*, ser. *Mathematica*, **40**, No. 4, 99.

Stoica, C., Mioc, V.: 1996 a, *Bull. Astron. Belgrade*, No. **154**, 1.

Stoica, C., Mioc, V.: 1996 b, *Rev. Roum. Sci. Tehn.*, sér. *Mécanique Appliquée*, **41**, 185.

Stoica, C., Mioc, V.: 1996 c, *Rom. Astron. J.*, **6**, 81.

Received on 7 July, 1997

NEAR-ESCAPE ORBITS IN MANEFF-TYPE PROBLEMS

CRISTINA STOICA[1], VASILE MIOC[2]

[1]*Institute for Gravitation and Space Sciences,*
Gravitational Research Laboratory,
Str. Mendeleev 21–25, 70168 Bucharest, Romania
[2]*Astronomical Institute of the Romanian Academy,*
Astronomical Observatory Cluj-Napoca,
Str. Cireşilor 19, 3400 Cluj-Napoca, Romania

Abstract. The Maneff-type problems (these associated to a potential function of the form $A/r + B/r^2$) are tackled again. Starting from the motion equations and first integrals in the space of McGehee's coordinates, one builds the infinity manifold and describes the flow on it.

Key words: celestial mechanics – qualitative analysis – Maneff-type problems.

1. INTRODUCTION

The global flow of the Maneff-type problems was obtained in both different phase planes (Stoica and Mioc 1996 a,b; Mioc and Stoica 1997) and the space of McGehee's coordinates (Diacu et al. 1996). In this last case, the collision manifold – resulted from the blow-up of the collision singularity, subsequent to the application of McGehee's (1974) transformations – plays an essential part. The reason is twofold: on the one hand, the phase space extends smoothly to the ex-singularity turned to a manifold; on the other hand, the flow of the collision manifold, although deprived of physical significance, offers rich informations about the behaviour of noncollisional orbits.

The study of another limit situation (physically speaking, because it however is not a singularity), the so-called infinity manifold is of importance, too (see, e.g., Craig et al. 1996). The flow on this manifold is as meaningless physically as that on the collision manifold is; nevertheless, it allows the understanding of the behaviour of near-escape orbits.

This paper will complete the extensive qualitative study performed in the

Rom. Astron. J., Vol. 7, No. 2, p. 183–188, Bucharest, 1997

above quoted papers (and not only; see Diacu et al. 1996 and the reference therein) on the Maneff-type problems, providing the description of the infinity manifold, as well as an insight about what happens with the bounded orbits neighbourhing infinity. In astronomical terms, we complete the study of near-parabolic orbits (by abuse of language) with the geometric description of the motion at very great distances from the field-generating body.

2. VECTOR FIELD AND FIRST INTEGRALS

The Maneff-type problem is associated to a central potential, hence it can be reduced to the central force problem featured by the potential function (see the quoted papers):

$$U(\mathbf{q}) = \frac{A}{|\mathbf{q}|} + \frac{B}{|\mathbf{q}|^2}, \tag{1}$$

where $\mathbf{q} = (q_1, q_2) \in \mathcal{R}^2$ is the configuration of the moving body (hereafter particle) with respect to the field-generating body (hereafter centre). Introducing the momentum vector $\mathbf{p} = \dot{\mathbf{q}}$, $\mathbf{p} = (p_1, p_2) \in \mathcal{R}^2$, the motion will be described by the system

$$\begin{cases} \dot{\mathbf{q}} = \mathbf{p} \\ \dot{\mathbf{p}} = \nabla U(\mathbf{q}), \end{cases} \tag{2}$$

which defines a vector field on the phase space $\mathbf{Q} \times \mathbf{P}$, where $\mathbf{Q} = \mathcal{R}^2 \setminus \{(0,0)\}$ and $\mathbf{P} = \mathcal{R}^2$ are the configuration space and the momentum space, respectively.

Equations (2), which explicitly read

$$\begin{cases} \dot{\mathbf{q}} = \mathbf{p} \\ \dot{\mathbf{p}} = -\left(\dfrac{A}{|\mathbf{q}|^3} + \dfrac{2B}{|\mathbf{q}|^4}\right)\mathbf{q}, \end{cases} \tag{3}$$

admit two first integrals: that of energy

$$|\mathbf{p}|^2 - \frac{2A}{|\mathbf{q}|} - \frac{2B}{|\mathbf{q}|^2} = h = \text{const.}, \tag{4}$$

and that of angular momentum

$$L(\mathbf{q}, \mathbf{p}) = q_1 p_2 - q_2 p_1 = C = \text{const.}, \tag{5}$$

where h and C are the constants of energy and angular momentum, respectively.

We resort now to McGehee's (1974) transformations of the second kind:

$$\begin{cases} r = |\mathbf{q}|, \\ \theta = \arctan\left(\dfrac{q_2}{q_1}\right), \\ u = \dot{r} = \dfrac{q_1p_1 + q_2p_2)}{|\mathbf{q}|}, \\ v = r\dot{\theta} = \dfrac{(q_1p_2 - q_2p_1)}{|\mathbf{q}|}, \end{cases} \tag{6}$$

$$\begin{cases} x = ru, \\ y = rv \end{cases} \tag{7}$$

$$\mathrm{d}t = r^2\mathrm{d}s \tag{8}$$

which all are real analytic diffeomorphisms. Under them, equations (3), (4) and (5) become respectively:

$$\begin{cases} r' = rx, \\ x' = r(hr + A), \\ \theta' = y, \\ y' = 0 \end{cases} \tag{9}$$

$$x^2 + y^2 = hr^2 + 2Ar + 2B \tag{10}$$

$$y = C \tag{11}$$

where, in (9), $' = \mathrm{d}/\mathrm{d}s$. These nonsingular equations and first integrals were used by Diacu et al. (1996) to construct the collision manifold. In what follows we shall start from equations (9)–(11) to tackle the infinity manifold.

3. INFINITY MANIFOLD

Let us put $\rho = 1/r$; formulae (9)–(11) become respectively:

$$\begin{cases} \rho' = -\rho x, \\ x' = \rho^{-2}(h + A\rho), \\ \theta' = y, \\ y' = 0 \end{cases} \tag{12}$$

$$\rho^2(x^2 + y^2) = h + 2A\rho + 2B\rho^2 \tag{13}$$

$$y = C \tag{14}$$

With the transformations

$$\begin{cases} \xi = x\rho, \\ \eta = y\rho \end{cases} \tag{15}$$

relations (12)–(14) read respectively:

$$\begin{cases} \rho' = -\xi, \\ \xi' = \rho^{-1}(h - \xi^2 + A\rho), \\ \theta' = \rho^{-1}\eta, \\ \eta' = -\rho^{-1}\xi\eta \end{cases} \tag{16}$$

$$\xi^2 + \eta^2 = h + 2A\rho + 2B\rho^2 \tag{17}$$

$$\eta = C\rho \tag{18}$$

Equations (16) exhibit a singularity at $\rho = 0$. This shortcoming can be removed via the rescaling of timelike variable

$$\mathrm{d}s = \rho \mathrm{d}\tau, \tag{19}$$

which leads (16) to

$$\begin{cases} \dfrac{\mathrm{d}\rho}{\mathrm{d}\tau} = -\rho\zeta, \\ \dfrac{\mathrm{d}\xi}{\mathrm{d}\tau} = \eta^2 - A\rho - 2B\rho^2, \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}\tau} = \eta, \\ \dfrac{\mathrm{d}\eta}{\mathrm{d}\tau} = -\xi\eta \end{cases} \tag{20}$$

where the energy integral (17) was used in the second equation (20).

Denote by M_∞ (infinity manifold) the intersection of the sets (invariant under the flow, as one can easily notice):

$$\left\{(\rho, \xi, \theta, \eta) \mid \xi^2 + \eta^2 = h + 2A\rho + 2B\rho^2\right\},$$
$$\left\{(\rho, \xi, \theta, \eta) \mid \rho = 0\right\}$$

namely

$$M_\infty = \left\{(\xi, \theta, \eta) \mid \xi^2 + \eta^2 = h,\ \theta \in S^1\right\} \tag{21}$$

is a 2-cylinder (or a 2-torus, because S^1 is the segment $[0, 2\pi]$ with the end points pasted together), all imbedded in the 4-space of the coordinates $(\rho, \xi, \theta, \eta)$. Of course, this is true only for $h > 0$, and M_∞ reduces to the circle ($\xi = 0, \eta = 0, \theta \in$ $\in S^1$) for $h = 0$, and $M_\infty = \emptyset$ (the particle cannot escape) for $h < 0$.

The vector field on M_∞ reads

$$\begin{cases} \dfrac{d\xi}{d\tau} = \eta^2, \\ \dfrac{d\theta}{d\tau} = \eta, \\ \dfrac{d\eta}{d\tau} = -\xi\eta \end{cases} \tag{22}$$

It is obvious that there are two circles of degenerate equilibria on the M_∞ torus: the upper circle UC ($\xi = \sqrt{h}, \theta \in S^1, \eta = 0$), and the lower circle LC ($\xi = -\sqrt{h}, \theta \in$ $\in S^1, \eta = 0$). The trend of all other orbits on M_∞ is easy to emphasize. Putting $\xi = \sqrt{h}\sin\psi$, $\eta = \sqrt{h}\cos\psi$, one is led to $d\psi/d\theta = -1$. Plotting the flow in the (θ, ψ)-plane, and taking into account (22), one observes that all orbits on M_∞ with $\eta \neq 0$ start from LC and end in UC. This means that, except for the two circles of equilibria, the flow on M_∞ consists of heteroclinic curves connecting LC with UC (Figure 1).

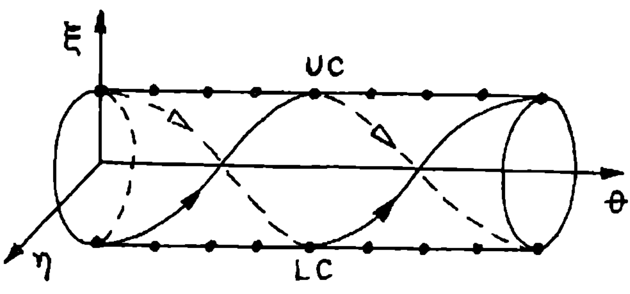

Fig. 1. – The flow on the cylinder M_∞.

4. REMARKS

The flow on M_∞ has no physical significance; however, due to the continuity with respect to initial conditions, it allows the understanding of the behaviour of trajectories which approach infinity (however without escape). It is worth noticing that, by (18), all orbits neighbour infinity in the region of UC (the escaping ones) or LC (those which come from infinity). Of course, if $h = 0$, UC and LC coincide (and M_∞ restricts to this single circle), whereas for $h < 0$ the particle cannot reach infinity (always bounded motion).

It is also clear that the coordinates (ξ, η) are quite the polar components of the velocity $(\dot{r}, r\dot{\theta})$. Moreover, the vector field (20) could be derived from the initial equations of motion. However, a new application of McGehee's technique (starting from (9)–(11)) seemed more elegant.

As a final remark, the time scale, (8), which governs the (physically meaningless) collision manifold, and the time scale (19), that features the (equally meaningless, from the physical standpoint) infinity manifold differ from each other.

All the above results offer a wider view and a deeper understanding as regards the orbits that approach inifinity in Maneff-type problems.

REFERENCES

Craig, S., Diacu, F., Lacomba, E.A., Perez, E.: 1996, *Preprint* **DMS-736-IR**, University of Victoria, Canada.

Diacu, F., Mioc, V., Stoica, C.: 1996, *Preprint* **DMS-755-IR**, University of Victoria, Canada.

McGehee, R.: 1974, *Invent. Math.*, **27**, 191.

Mioc, V., Stoica, C.: 1997, *Baltic Astron.*, **6**, 637.

Stoica, C., Mioc, V.: 1996a, *Bull. Astron. Belgrade,* No. **154**, 1.

Stoica, C., Mioc, V.: 1996b, *Rom. Astron. J*, **6**, 81.

Received on 7 July, 1997

THE SINGLE IMAGE MODERNISED ASTROLABE OF BUCHAREST OBSERVATORY

PETRE POPESCU, RADU POPESCU, PETRE PARASCHIV

Astronomical Institute of the Romanian Academy,
Str. Cuţitul de Argint 5, 75212 Bucharest 28, Romania

Abstract. In Bucharest was installed a Danjon astrolabe, transferred from the Royal Observatory of Brussels. This paper deals with the improvements brought to this instrument in order to gain stability, observation accuracy and efficiency. The changing of the transparent prism with a Zerodur reflecting one, the new dual task of this astronomical instrument (self-collimating system and star tracker) and the acquisition of images using a CCD camera are the technical solutions of the modernisation project.

Key words: astrolabe, CCD observations.

1. SCIENTIFIC PROGRAMS APPROACHABLE BY MEANS OF THE DANJON ASTROLABE

The astrolabe as conceived by Danjon was until the end of the eighties one of the most remarkable astronomical instruments, able to observe the stars, planets and their satellites, and more recently the Sun (e.g. Débarbat & Guinot 1970).

Being able to observe both stars and solar system bodies, it can be actually helpful in versatile observations, to which the present high precision techniques have no access. It is worth presenting briefly some interesting domains of research where the astrolabe is very useful (Chollet 1995):

a) The link of reference systems

The versatility of the instrument makes possible the link between conventional and dynamical systems by observations of planets and their satellites, on the one hand, and of stellar (inside and outside the Galaxy) on the other. By systematic radiostars observations one can link the conventional inertial system with that of extragalactic radiosources.

b) Variations of local vertex

The Earth crust produces periodic movements with regard to the terrestrial reference frame: the total amplitude is of about 0".01, and the dominant frequencies are diurnal and semidiurnal. By observations of bodies situated in and outside the Solar System, the variations of the local vertex can be deduced.

c) The mapping of stellar catalogues

The most modern stellar catalogues (FK5 and HIPPARCOS) contain very precise positions of stars; the disadvantage is that these positions are subjected to time degradation. The effect of proper motions in producing that damage is well known. In the absence of other space techniques or new missions such as HIPPARCOS or TYCHO, the ground-based observations with improved techniques still remain requisite.

d) The measurement of the Solar position and diameter.

Only the astrolabe, specially adapted for this kind of observations, can measure the position of the Sun with an accuracy better than 0".4 (for one observation). The same precision can be reached in measurements of the Solar diameter. These programs permit to verify the theories of the Earth movements and also to connect the dynamic reference system to the stellar system.

2. QUALITIES AND FLAWS OF DANJON ASTROLABE

A brief analysis concerning the technical possibilities of visual astrolabes in comparison with the modern techniques emphasized a complex of qualities and flaws that persuaded the international astronomical community to continue the research in order to improve the astrolabe. Among the qualities we can quote the stability of the instrumental reference system due to the firm compact prism and the mercury pool. The HIPPARCOS telescope is in fact an astrolabe without the mercury pool, which reaches an observational accuracy of about 0".02 (for one observation). This accuracy can be attained also by improved ground-based observations.

The flaws of Danjon astrolabe are the following:

a) The poor luminosity. The use of a glass prism that works in the transmission mode, and of the bi-refringent Wollaston micrometer leads to a loss of luminosity. The aperture of the objective is divided into two parts, each of them being equivalent to a 60 mm aperture; since the mercury only has a reflectivity of 70–75% (Xu 1992), its half lens works as an aperture of 45 mm. This fact leads also to an even more severe loss of luminosity in the case of the reflected image. The performed observations are visual (the eye's integration time is of about $0^s.2$). That is the why the greatest magnitude is limited to 6.5.

b) The variation of the prism angle caused by the thermal instability of glass (due to the thermal expansion coefficient for optical glasses).

3. SOLUTIONS ADOPTED IN ORDER TO IMPROVE THE QUALITIES OF THE DANJON ASTROLABE

a) The construction of the Zerodur-reflecting prism

The change of the transparent prism with a reflecting one (Popescu et al. 1996), cut from a block of Zerodur, was an important improvement of the astrolabe. The shape of the prism (based on an idea of F. Chollet) was detailed until February 1996 and the manufacturing was performed by the Romanian Optical Industry (I.O.R.). The system contains two prisms glued at the contact point:

– the first (45°) is used together with the left pupil and the mercury bath as an autocollimation system,

– the second (67° 30') serves to reflect the light emitted by celestial bodies located at a 45° zenith angle.

The plane surface tests were performed in the I.O.R. laboratory with a V-100 PHASE interferometer (λ = 633 nm). The results are given in Table 1.

Table 1

Prism	Code, utility of the surface and position	Planeity
I	autocollimation on the mercury pool; tilted at 45°.	0.092λ
I	autocollimation for positioning the prism with regard to the astrolabe; vertical.	0.11λ
II	observation of stars; tilted at 67°.5.	0.13λ

To obtain a precise determination of prism angles, measurements were made with a Moller goniometer (the reading error is of ±1"). The results are given in Table 2.

Angle	Result
45°	44°59′59″.3
67°30′	67°21′56″

The piramidity is of 30" ± 5. The mounting of the optical system was conceived according to the mechanical modification of the astrolabe.

b) The replacement of the Wollaston micrometer with a LED lighted reticle.

By removing the Wollaston, the focal length of the astrolabe varied, so we had to shift the image transporter backward versus the eyepiece (and at the same time versus the CCD camera) in order to avoid parallax errors. In the place of the Wollaston we put the illumination system and the reticle. The reticle serves for instrumental constants determinations. It is made from a very thin thread of spider cocoon (10 μm).

The angle between the vertical and horizontal threads is of $90^\circ \pm 0^\circ.2$.

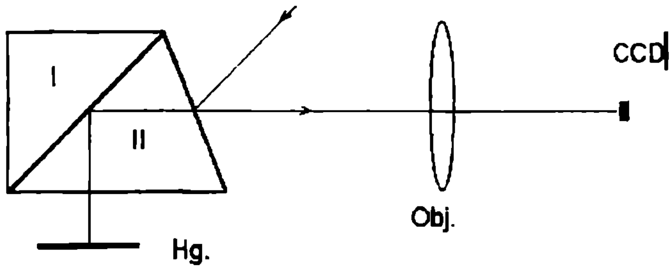

Fig. 1. – The optical scheme of the modified astrolabe.

c) The automatic positioning in azimuth.

A gear demultiplication system (1/170) was used in combination with a step motor driven by a microcontroller MC68HC11 (half step mode). For the time being the system works in an open loop but we soon intend to close the loop by an ENCODER system.

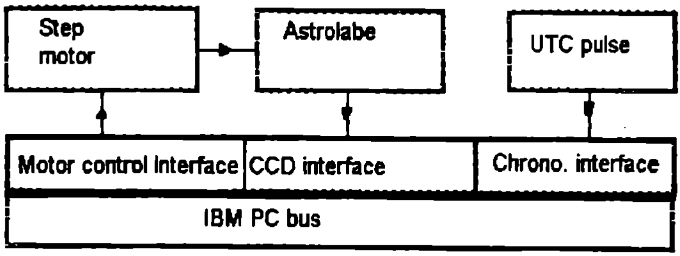

Fig. 2. – The electronic scheme of the modified astrolabe.

d) The use of a CCD camera as star tracker and autocollimation system.

The camera used is a COHU 4710 (borrowed from Paris Observatory, DANOF) that works in frame transfer mode. It reads in the CCIR video standard (25 images by second). Two frames (20 ms) are necessary to make an image (40 ms). With the aid of a CICLOP plate the video signal is digitized on 8 bits. The plate also transforms the real camera in a virtual one, having the dimension of 512 x 512 pixels. The dimensions of a pixel are of $8.5\mu m \times 10.5\mu m$.

4. THE PROCESSING OF IMAGES OBTAINED WITH THE COHU – CCD CAMERA

As we wrote in the description of the system, COHU is a TV camera. The integration time for one frame (interlaced mode) is of 40 ms. In order to transfer one image from the interface plate of the camera to the hard-disk the processor takes on the average over 100 ms. For this reason we adopted a time interval of image acquisition of 200 ms. The software conceived in Turbo Pascal offers the possibility of working with windows of 100×512 pixels that can be shifted in any direction at different velocities (electronic star tracker).

A file with a capacity of less than 64 Ko contains the intensities of the 100×512 pixels digitized on 8 bits, the time of acquisition (by means of time interface), and the azimuth of observation.

4.1. THE CALCULUS OF HORIZONTAL AND VERTICAL TILT OF THE ASTROLABE

By using the reflecting prism of 45° angle, the mercury pool, and the CCD camera, the new astrolabe becomes a very accurate autocollimatic instrument. Because of its 3500 mm focal length, the accuracy of the tilt angle calculation is of about 10 mas.

The algorithm is a very simple one:

a) A window centered on the illuminated reticle is chosen. By removing the mercury pool, only the image of the reticle is "viewed" by the camera (Fig. 3). The center determination of horizontal and vertical reticles is performed by the least squares method. The centering of the reticles can be done by several methods (Gaussian profile, mediane, gravity center method, inflexion point method). We chose the last two methods.

b) By using the mercury pool the new frame contains the reticle and its image reflected in mercury. Because of the poor reflection coefficient of the mercury (of about 75%) and of the multiple reflections inside the astrolabe, the intensity loss of the autocollimated reticle is of about 50% (Fig. 4). The subtraction of the first frame from the second is operated, then the algorithm is repeated as at point a), with the difference that, instead of the reticle, now we work with a reflected image (Fig. 5).

c) The half distance between the two horizontal reticles is in fact the true horizon. It is done in a first approximation by the equation $Y = \mathrm{Lgn}$, where Lgn is the row number in the CCD matrix.

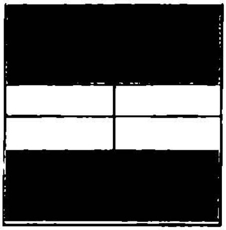

Fig. 3.
The direct image of the reticle.

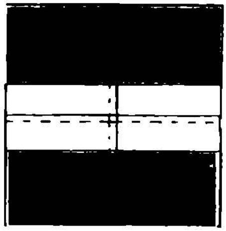

Fig. 4.
The direct and reflected image of the reticle.

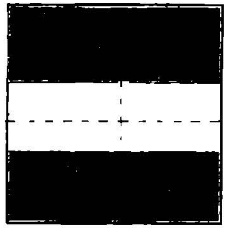

Fig. 5.
The reflected image of the reticle.

4.2. THE STARLIKE IMAGES ACQUISITION MODE

As we already said, the integration time is of 40 ms. The astrolabe "half-pupil" aperture is of about 60 mm. In these conditions, according to Hou (1995), the seeing "is frozen". The seeing parameters are given in Table 3.

On the other hand, the number of photons that fall on the CCD is in this case very much diminished. From experience we can say that in the conditions of a S/N rate equal to 2–3, the greatest magnitude reached is mv = 4.0 (for an exposure of 40 ms). The rate of S/N can be improved by using a technique called CINE-CCD mode (Fort et al. 1984).

Table 3

Summary of the parameters for the atmospheric turbulence

Parameter	**Expression**	**Typical value**
Fried parameter r_0		10–30 cm
Coherence time τ		5–30 ms
r.m.s. image motion σ_m	$\sigma_m^2 = 0.358 \cdot \left(\frac{\lambda}{D}\right)^{\frac{1}{3}} \cdot \left(\frac{\lambda}{r_0}\right)^{\frac{5}{3}}$	for $\lambda = 6\ \mu m$, $D = 10$ cm, $\sigma_m = 3.59 \times 10^{-5}$

In opposition with the TDI (time delay integration) mode, this image acquisition mode works with large series of images of a very short exposure time, any sequence being compared with a movie sequence. The sum of all these images is equivalent with a long time integration image. If the system noise is very low and uncorrelated in time and space, then the signal to noise rate (S/N) can be improved by adding all the images with the square root of the number of images. Depending on the existence of some reference points in the grabbed image, there are several summing algorithms:

- If there exist stars bright enough to provide an S/N greater than 3, for centering the frames, the origin of Cartesian axes system is chosen in the centroid of stars (fitted by a Gaussian mask);

- If there are not any bright stars in the field, the autocorrelation function between two consecutive images is one solution; another solution is the correlation between the image and a starlike PSF filter.

The star's passage time across the field of the astrolabe is given by the formula:

$$t \leqslant \min \{22^s \sin A \cdot \cos\varphi;\ 22^s(\sin\varphi + \cos\varphi \cdot \cot z \cdot \cos A)\} \tag{1}$$

where: A is the azimuth measured from South,

φ is the latitude (in the case of Bucharest 44° 24' 50."38),

z is the zenithal distance.

The rate transfer from the camera to the hard-disk is of about 200 ms so that the number of images taken in the star's passing interval is great enough to improve the S/N rate and to build-up the star's trajectory. Knowing in the first approximation the azimuth position, the parallactic angle, and the zenital distance of a star we can schedule a vertical movement. The motion velocity is:

$$v_y = 15 \cos\varphi \sin A, \tag{2}$$

being measured in ["/s].

In this way the star remains in the same position on the vertical and moves only on the horizontal with the velocity:

$$v_x = 15(\sin\varphi + \cos\varphi \cot z \cos A) = 15\frac{\cos\delta \cos S}{\sin z} \tag{3}$$

where, δ is star's declination and S the parallactic angle.

These assumptions were made neglecting the r.m.s. image motion due to the atmosphere. We can conclude that in a first approximation the camera behaves as a star tracker.

4.3. THE TRACKING OF A STAR

Because the field is narrow of about 5'.5, a star trajectory can be approximated by a straight line. According to Chollet (1996), we can build up the trajectory of a star from the position in the frame at the moment recorded by the time interface. If the angle between the two reference systems (CCD system and local vertical system) is zero, it means the CCD rows are perpendicular to the horizon, and we can approximate the motion equations as follows:

$$y = \Delta t \cos\varphi \sin A + \epsilon_y$$
$$x = \frac{\Delta t \cos\delta \cos S}{\sin z} + \epsilon_x \tag{4}$$

We denote:

$$\alpha_1 = \cos\varphi \sin A,$$
$$\alpha_0 = S, \tag{5}$$
$$\cos\delta \cos S = \frac{\alpha_1}{\tan\alpha_0}$$

Because of the mirror reflection, the expression of y shifts its sign:

$$\cos\varphi \sin A < 0 \quad \text{to East and} \quad \cos\varphi \sin A > 0 \text{ to West.} \tag{6}$$

But,

$$\frac{\Delta z}{\Delta t} > 0 \text{ to East and } \frac{\Delta z}{\Delta t} < 0 \text{ to West.} \tag{7}$$

The motion equations in the two directions become:

$$y = \alpha_1 \Delta t + \epsilon_y$$
$$x = \frac{\alpha_1}{\tan\alpha_0}\Delta t + \epsilon_x \tag{8}$$

When the CCD reference system (XOY) is tilted at angle β with respect to the local vertical system (xOy), the motion equations are given in both systems:

$$\text{(CCD)} \qquad \begin{aligned} X &= x\cos\beta + y\sin\beta \\ Y &= -\sin\beta + y\cos\beta \end{aligned} \tag{9}$$

$$\text{(horizon)} \qquad \begin{aligned} x &= X\cos\beta - Y\sin\beta \\ y &= X\sin\beta + Y\cos\beta \end{aligned} \tag{10}$$

which means:

$$\begin{aligned} X &= \frac{\alpha_1 \cdot \cos(\alpha_0+\beta)}{\sin\alpha_0}\Delta t + \epsilon_x\cos\beta + \epsilon_y\sin\beta \\ Y &= \frac{\alpha_1 \cdot \sin(\alpha_0+\beta)}{\sin\alpha_0}\Delta t - \epsilon_x\sin\beta + \epsilon_y\cos\beta \end{aligned} \tag{11}$$

With the help of the least squares method one can determine the coefficients $\beta, \epsilon_x, \epsilon_y$. The star trajectory can be described in the first approximation as a function

$$Y = Y(X).$$

The passage moment of the star at the zenital distance given by the prism geometry can be deduced by solving the equation:

$$Y(X) = \text{Lgn}. \tag{12}$$

5. CONCLUSIONS

The first aim of this project is to obtain a very accurate astrometric instrument. For the time being, we are not as much interested in observing faint stars as in determining all the inner parameters of the new astrolabe. The next step is the replacement of this camera with an Intensified CCD camera. That camera must have direct access to pixels, variable acquisition time (up to 200 ms), and high data transfer rate. Because it is impossible to use the TDI mode for astrolabe observations, we think that the CINE-CCD mode remains the best solution.

REFERENCES

Chollet, F.: 1995, *Astrolabes with CCD*, private communication.

Chollet, F.: 1996, *Deduction of Sun diameter by CCD observations*, private communication.

Débarbat, S., Guinot, B.: 1970, *La metode des hauteurs egales en astronomie*, Gordon & Breach (ed.), Paris.

Fort, B., Picat, J.P., Cailloux, M., Mauron, N., Dreux, M., Fauconnier. T.: 1984, Astron. & Astrophys. **135**, 356.

Hou, J.: 1996, in N. Kaifu (ed.), *Ground-Based Astronomy in Asia*, Proc. 3rd East-Asian Meeting on Astronomy, Tokyo, July 17-21, 1995, National Astron. Obs., Tokyo, 533.

Popescu, P., Popescu, R., Paraschiv, P.: 1996, Communication held at the Annual Scientific Session of AIRA, Bucharest.

Xu, J., Wang, H., Zhang, Z.: 1992, in *Developments in astrometry and their impact on astrophysics and geodynamics*, I. Mueller & B. Kolaczeck (eds.), Proc. IAU Symposium No. **156**, Shanghai, China, 89.

Received on 11 February, 1997

ASTROMETRIC POSITIONS OF THE COMET 1995/O1 (HALE-BOPP)

GHEORGHE BOCŞA, MIREL BÎRLAN

Astronomical Institute of the Romanian Academy,
Str. Cuţitul de Argint, 5, 75212 Bucharest 28,
Romania

Abstract. Precise positions of the comet 1995/O1 (Hale-Bopp) observed in Bucharest are presented. These observations were made with a 380/6000 mm astrograph. We used astrophotographic plates and an Ascorecord measuring machine.

Key words: comet, astrometry.

The comet 1995/O1, very well known by the astronomers (professionals and amateurs) as the comet *Hale-Bopp*, offered us a brilliant cosmic show. The computed preliminary orbit reveals that the comet 1995/O1 is for the first time when it passes across the Sun. Systematic observations of its "atmosphere" allow the identification of new chemical compounds (Circular 6568, 6573, 6614, 6625, 6631) never before reported for any other comet.

During autumn 1996, several nights of observation allowed us to perform precise astrometric positions of the comet. The observations were made with the *Mertz-Prin* astrograph (380/6000 mm) from Bucharest.

These positions were calculated using 5–10 PPM reference stars, chosen around the comet, not farther than 1° from the center of the plates. From the α_i, δ_i coordinates (corrected with the proper motions) of the reference stars, and α_0, δ_0 of the center of the plate, the normal coordinates X_i, Y_i were computed. Both Turner's (constants) and Schlesinger's (dependences) methods (Brouwer & Clemence 1961) were used to compute the normal coordinates of the comet. Then, starting with the normal coordinates X, Y, the topocentric coordinates α, δ, of the comet were determined.

The results are presented in Table 1. The first column contains the date of the observations (year, month, day with fraction of day); the topocentric right ascension and the declination of the comet for the 2000.0 epoch are presented in the second and the third column, respectively; the last column contains the number of the reference stars used for computing.

Table 1

Astrometric positions of the comet 1995/O1 (Hale-Bopp)

DATE		UT	$\alpha_{2000.0}$	$\delta_{2000.0}$	N
1996	09	10.76657	$17^h33^m34^s.69$	$-6°00'26''.0$	5
"	09	10.77453	17 33 34 .45	−6 00 24 .3	5
"	09	10.73788	17 33 10 .43	−5 57 19 .0	10
"	09	11.75034	17 33 10 .10	−5 57 16 .7	10
"	10	03.71416	17 29 51 .17	−4 52 15 .2	8
"	10	15.69075	17 32 18 .87	−4 16 34 .4	9
"	10	15.69802	17 32 18 .99	−4 16 33 .3	9
"	11	04.67492	17 42 05 .11	−3 04 25 .2	10
"	11	04.68219	17 42 05 .34	−3 04 22 .6	10
"	11	12.66901	17 47 44 .94	−2 27 43 .1	7
"	11	12.67697	17 47 45 .29	−2 27 42 .8	7

These astrometric observations were already reported to the Central Bureau for Telegrams of the International Astronomical Union and used to improve the orbital elements of the comet.

REFERENCES

Brouwer, D., Clemence, G.: 1961, *Methods of Celestial Mechanics*, Academic Press, New York & London.

* * * *Circula Nos. 6568, 6573, 6614, 6625, 6631*, Central Bureau for Astronomical Telegrams of the International Astronomical Union.

Received on 8 July, 1997

CONSTANTIN GOGU
(1854–1897)

A century ago C.I. Istrati accompanied Constantin Gogu at the latter one's funeral with the words "let his memory be forever respected and never forgotten".

This is exactly what we, their successors, are also trying to do, whenever we think of our illustrious predecessors. And Constantin Gogu was one of them.

He was born on 30 May, 1854, at Câmpulung Muscel in a family with evident inclinations for mathematics: thus, both his brothers from his mother's two marriages (Gogu and Ianculescu) pursued mathematical studies. One of Gogu's brothers took his degree in mathematics in Bucharest, while a sister by the same father died before taking her degree in the same science. Another brother, Mihai Ianculescu, was a lecturer at the Bucharest University.

This atmosphere of study had a strong influence on Constantin Gogu even beginning with the years of his childhood: after he had finished the first elementary classes in his native town, he became one of the permanent prize-winners of the *Matei Basarab* high school in Bucharest (1866-1873). Afterwards, he enlisted for the Faculty of Sciences in Bucharest, the section of mathematics. Later, on he taught mathematics at the *School of Trade* (1 September, 1876–1 September, 1877).

In October 1877 he went to Sorbone with a study scholarship. There he took his degree in mathematics (31 July, 1878). Afterwards, he became "a free student" of the Astronomy School of Paris Observatory, until 1881. Even beginning with that period Gogu had already approached the study of the Moon, which was to be also the subject of his doctoral thesis. In the *Rapport annuel de l'Observatoire de Paris pour 1881* (p. 19) it is mentioned: "Mr. Gogu, a Romanian astronomer student, of our Observatory, left us last November to be appointed professor at the University in Bucharest; he published in Volume XVII of our Memories an important work on the inequality of the Moon produced by the planet Mars".

After his return home, he became a professor at the *School of Artillery* and *Engineer Officers* in Bucharest, and also a professor of analytic geometry at the *Faculty of Sciences*, at the *School of Architecture*, as well as at the *Nifon Seminary* (he had even been appointed guardian of the *Nifon* Institution, as his mother, born Rusescu, was a niece of Nifon, the metropolitan bishop of Hungary-Wallachia).

He returned again to Paris to present his doctorate at the Faculty of Sciences; on 7 February, 1882 Gogu became the third Romanian to defend a thesis on mathematics

Rom. Astron. J., Vol. 7, No. 2, p. 201–204, Bucharest, 1997

and respectively the second one on celestial mechanics. The title of the thesis was *Sur une inégalité lunaire à longue période due à l'action perturbatrice de Mars et dépendant de l'argument* $\omega + l - 24l' + 20l''$. The examination commission was made up of Briot (replacing Puiseux as President), Bouquet and Tisserand. The thesis was dedicated to the Rear admiral E. Mouchez, the director of the *Observatory in Paris* and member of the *Institute* and of the *Bureau of Longitudes*, to Dimitrie Petrescu, a professor at Bucharest University and also a member of the Standing Committee of Public Instruction, as well as to his main teachers, V. Puiseux and F. Tisserand.

The thesis was mainly elaborated on the basis of *Delaunay's theory of lunar motion*, as well as of some articles of the English astronomer *Neison*, published in *Monthly Notices*. Gogu planned to study for the lunar motion the presence of some long-periodic inequalities, produced by other bodies than the Sun, such as the closest planets to us — Venus, Mars and Jupiter. The idea was not new; Hansen had already introduced in the *Lunar Tables* two long-periodic inequalities due to the perturbing action of Venus (inequalities also verified by Delaunay and Tisserand).

Neison had pointed out some long-periodic inequalities due to the planets, too. According to his computations, the perturbing effect of Mars could lead to a term with a period of 406 years, an inequality that was too important not to have attracted the attention of the astronomers and also, too great, taking into account the relatively negligible mass of Mars with respect to that of the Sun.

By contrast, C. Gogu took into consideration the variations of the elliptic elements of the Moon due to the perturbing action of the Sun, going as far as the second degree terms as against the average motions of the Earth and of the Moon (the lunar orbital elements could not be considered constant even in a first approximation). He also took into account the inclination of Mars orbit with respect to the ecliptic. Considering also the terms which had been neglected by Neison, Gogu demonstrated that, for the respective perturbation, the coefficient was practically negligible, contrary to the facts shown by Neison. Indeed, the inequality established by Gogu was:

$$0'', 00034 \sin(l - 24l' + 20l'' + 176°27', 3),$$

in comparison with approximatively $7''$, term established by Neison.

Gogu ended his 101 pages of analyses and laborious calculations (almost 500 operations) with the following conclusion: "*... in the two long period lunar inequalities due to the perturbing action of Venus, discovered by Mr. Hansen and estimated by Mr. Delaunay (Connaissance des Temps, the years 1862 and 1863), the terms that contribute almost entirely to the formation of the coefficients of the two inequalities are those containing as factor* $\sin \gamma''$ *multiplied with half of the perturbing planet orbit inclination, with respect to the ecliptic plane; likewise, also the coefficients* A_5 *and* A_6 *of the analogous terms have, for the inequality we are dealing with, the highest values; however, in our case, the factor* γ'' *by which they are multiplied, is much smaller than* e' *and* e''*, which leads to the conclusion that in the end those are not the highest value terms*".

Gogu's thesis did not remain without echo. Adams, who had discovered, together with Le Verrier, the planet Neptun, urged him to publish an abstract of his conclusions in

Memoirs of the Royal Astronomical Society. The article was accompanied by a letter in which he underlined the fact that the difference with regard to Neison's results were due to the same calculus errors of the latter one.

But there were also other opinions regarding Gogu's work. Teofil Vescan appreciated Gogu's thesis even as a *"work that was less than modest and completely anachronistic today"* (*Tribuna*, 23 August 1939). The truth was that presented by his professor, Petre Sergescu, a specialist in the history of mathematics, in a retort published three weeks later by the same newspaper (Tribuna, 11 September 1939). He showed that Gogu's thesis "raised a great echo at its time, as it abolished the theory of the English astronomer Neison concerning the inequalities of the Moon's average longitude".

Sergescu was right, too: otherwise how could we explain the fact that Radau included in the table published in *Annales de l'Observatoire de Paris, Mémoires*, t. XXI (R. Radau: *Sur les inégalités planétaires du mouvement de la Lune*) arguments of relatively short period, which do not contain lunar elements, but only the L' and L'' longitudes ? Their importance comes especially from the indirect action exerted by the planets on the Moon, from perturbations on the Earth. In the 7^{th} line of the table also used by Tisserand, the fifth order coefficients were mentioned as belonging to two authors, Neison and Gogu (*Cours de Mécanique céleste*, Vol. III, 1894, p. 379). Gogu's result is also quoted in *Encyclopedie des Mathematischen Wissenschaften*, Vol. VI, 1915, p. 683.

The fact that Gogu was considered by Tisserand as a first class specialist results also from the fact that Tisserand consulted with Gogu on the method errors that appeared in the works of the American astronomer Stockwell (published in *Sillimman's American Journal — 1880*) from *Théorie du mouvement de la Lune* by Delaunay, as well as in the works of other authors, such as Laplace, Pontécoulant, Giovanni Plana. All these had taken into consideration, for the Moon's longitude, terms due to the perturbing action of the Sun that did not disappear if the Sun's mass were considered null. These were annulled at the same time with the Sun's mass, as Gogu later showed in the memoir published in the *Annales de l'Observatoire de Paris*, in 1884. It was in the same memoir that Gogu showed that the difference between Hill and Delaunay concerning the integrals of the lunar motion equation came from different versions on the integration constants. He once more confirmed Delaunay's scientific rigour, from which he claimed he had actually started his thesis.

In the *Annales de l'Observatoire de Paris* it is recorded that: *"Mr. Gogu, a free Romanian student of our school, has successfully passed the doctorate exams in mathematical sciences and has left the Paris Observatory"*.

In response to an article by Paul Tanco, Gogu published five letters in *Scientific Recreations "on the establishing of the Easter celebration"*. It actually concerned also the Moon's motion ; the Easter date depends on the date of the full Moon, which can take place on the day of the vernal equinox or afterwards. In the 4^{th} century, when the council of Niceea took place (325 A.D), it was believed that a lunar phase repeated itself exactly every 19 years; thus, a one day errors was introduced every 308 years, namely a difference between the astronomical paschal Moon and that established by the Orthodox Church. This error had been eliminated from the calendar of the Western Church through the reform imposed by the Pope Gregory the XIIIth. There obviously remained small errors due to the difference between the true and mean positions of the Moon. Gogu approached thus a problem tackled before by another great mathematician, Emanuel Bacaloglu, at a meeting of the Academy (the *Annals of the Academy*, 2nd series, Vol. II, *About the Calendar*, 20 March/1 April 1880), a reception speech at the Academy, in which he sustained the superiority of the Gregorian Calendar and the necessity that it should be introduced in all cultivated countries. (Through the decree-law of 5/18 March 1919, the Gregorian Calendar, or the *New Style*, was to be adopted also in our country.) Gogu underlined once more the scientific character of the calendar reform.

After his return home following the defence of his doctoral thesis, Gogu participated together with David Emanuel in the contest for the chair of analytic geometry. Although Spiru Haret, a member in the examination commission, had preferred David Emanuel, Gogu was declared the successful candidate, to fill the chair beginning with 28 September 1882 and until the end of his days (the chair of celestial mechanics was taken up at that time even by Gogu's professor, Dimitrie Petrescu). Gogu was to be followed at that chair by the student who had very consciously lithographed the lecture taught, namely by Gheorghe Ţiţeica.

In parallel with the lectures at the University, Gogu also taught at the School of Bridges and Highways Construction from 1887 until 1888, when the new law of holding more than one office determined him to make an option for the first chair.

He was a good professor, who imposed on his students order and intelectual discipline, but also love for science.

This is also proved by the fact that he made efforts to set up the society "The Friends of Mathematical Sciences" (24 January, 1894), by the way in which he encouraged the appearance of the *Mathematical Gazette*, as well as by the support he gave to the society "The Romanian Youth", meant to stimulate the competition between high-schools to increase the prestige of Romanian science. He was the first president of the "Romanian Society of Sciences" (13-30 January, 1897).

His scientific merits were acknowledged by the Romanian Academy, which included him among his corresponding members in the year 1889. On that occasion he published (in the Volumes 13 and 14 of the Academy) two works: *On the curvature radius of the curve born by a point invariably connected to a circle that rotates, without shipping, on a fixed straight line* and *On the variation of gravity in the same locality.*

Unfortunately, a cruel disease put an end to his life, while still at full creative power. He died in Craiova at the age of even less than 43 years old; on 31 January, 1897, he was buried at Câmpulung, beside his family.

Magdalena Stavinschi

ANNUAL CONFERENCE ON ASTRONOMY

The Annual Conference of Astronomy, organized by the Astronomical Institute of the Romanian Academy together with the Romanian National Committee of Astronomy, took place in Bucharest in 17 and 18 April 1997.

The conference was dedicated to the commemoration of some illustrious representatives of astronomy from our country, as well as to the presentation of the latest astronomical results of the Romanian researchers.

Following the opening speech of the Academician Romulus Cristescu, the President of the Mathematical Sciences Department of the Romanian Academy, and of a presentation by Dr. Magdalena Stavinschi, the Director of the Astronomical Institute of the Romanian Academy, on the situation of astronomy in Romania in the 90th year of existence of the Astronomical Observatory in Bucharest, Prof. Arpad Pal evoked the personality of Academician Constantin Drâmbă, a remarkable specialist in the field of celestial mechanics and former director of the above mentioned institute in the period from 1963 to 1977, deceased on 10 February 1997.

Although 20 years have elapsed since the passing away of our Professor, the Post Mortem Academician Călin Popovici, the evocative words of Dr. Emilia Ţifrea brought him back to our minds, together with his results and innovative ideas in the field of Astrophysics and Space Research launched in our country and at the international level.

On the same occasion Dr. Magdalena Stavinschi presented the life and activity of Constantin Gogu, a corresponding member of the Romanian Academy, on the commemoration of 100 years since his death.

In view of an astronomical event to take place at the end of this century, of a special importance for our country, namely the total solar eclipse of 11 August 1999, a series of papers on special programs for the observation of this eclipse, as well as on the stage of the researches related to the integral coronal radiation, coronal mass ejections, polar plumes and complex solar filaments were presented; these researches can be improved by new results obtained through the observations that will be possible during the total solar eclipse of August 1999.

The situation of the celestial bodies visible during the eclipse and the calculation method of their ephemerides were also presented.

In 1997, the Astronomical Institute participated in the PHESAT program, in collaboration with researchers from the Bureau of Longitudes in Paris, obtaining new results of the observations on Saturn's satellites made in Bucharest.

The results of the determinations of some physical parameters of the asteroids, through CCD photometry, obtained also in collaboration with French researchers from the Observatory of Paris-Meudon, were also presented.

The latest results on the improvement of stellar positions by means of the global reduction methods, preliminary results obtained with the modernized Danjon astrolabe, as well as these concerning the dynamics of the artificial satellites of type GPS and Cosmos were presented, too.

The researches of stellar photometry, in the field of variable stars, produced papers in all the three observatories of the Institute, namely from Bucharest, Cluj and Timişoara, such as: the calculus of the line profiles with the interacting binaries, the long term variation of the DY Pegasi pulsating star period, the complex modulation of the orbital period of the U Coronae Borealis binary, minima determinations and appreciations on the modification of the period of the variable with 44i Bootis eclipse, analysis of the variation of the pulsating star of type Cephei — BW Vulpeculae.

Interesting results were also obtained in the large-scale study of the Universe, through N-body simulations of the interactions in the galaxy clusters.

Celestial mechanics was present in the papers referring to the structure of the phase space in the Schwarzschild problem and to the KAM theory applied to the Gylden problem.

At the end there were debates on the perspectives of teaching astronomy in the college, as well as on the contributions of some centenary journals to the presentation of some astronomical topics for the general public.

Maria Magdalena Cârşmaru

"THE OPEN DAY" AT THE ASTRONOMICAL INSTITUTE

The Astronomical Institute of the Romanian Academy is the first research centre in Romania that decided to open its doors for the general public once a year. The day chosen for the inauguration of this manifestation was Saturday, October 11, 1997. Thousand visitors roughly had in this way the possibility to know more intimately the preoccupations, the efforts, and the results of a research institute.

Along eight hours the whole staff of the Bucharest Observatory of the Institute was at the public's disposal, guiding the groups of visitors through halls and domes, providing explanations and answering endless questions. The people showed a high interest for both such a special research and the most important astronomical events the researchers focus on, mainly the so much expected total solar eclipse of 1999. There were presented the instruments, as well as posters intended to offer an eloquent image of the difficulties and achievements the astronomical work faces. The scientific movies and the computer simulations constituted a special attraction.

In this way the general public had the opportunity to know the manner in which the money invested in scientific research is used, as well as many achievements of the Romanian astronomy, often better known abroad than in our country.

We hope that the initiative of the Astronomical Institute will be followed by other research institutes in Romania.

Magdalena Stavinschi

Tiparul executat la Tipografia STELIAN-București
Tel./Fax: 01.250.22.81.

NOTICE TO AUTHORS

ROMANIAN ASTRONOMICAL JOURNAL is a journal which appears twice a year and is open to original contributions in Astronomy and related disciplines. The contributions – in English or French – can be accepted only if they were neither published before nor destined to any other publication.

Manuscripts should be submitted in duplicate; they must be typewritten on white A4-sized paper, onesided and doublespaced (enclosing abstract, references, footnotes and figure captions). The first page should contain: the article's title (brief and informative), author's name and affiliation, followed by an Abstract in English and Key words. The text should be clear and concise (it is recommended not to exceed 10 pages). *The Abstract* will present clearly the principal conclusions of the work, in no more than 10–15 lines.

Chapters and Paragraphs. Papers, except short notes, should be divided into Chapters, numbered by Arabic numerals. Chapters may be divided into Paragraphs denoted by the number of the Chapter and the number of the Paragraph; each Chapter and each Paragraph should have a short descriptive title (e.g. "3.2. Results").

Formulae have to be numbered consecutively in Arabic numerals, too, but included in parenthesis on the right side of the manuscript; all formulae should be written in legible form. The author should underline, in the text and in formulae, with lead pencil, all characters which he wants to be italics; the bold appearance will be indicated by double underline.

Tables should be numbered consecutively in Arabic numerals; each should be typed on a separate sheet.

Figures and Illustrations should be submitted separately in such a form as to permit reproduction without retouching. Any lettering should be large enough to be legible after the figure has been reduced in size for printing. Captions should be given on a separate sheet and labelled to show which illustration each is accompanying. All the figures should be numbered consecutively in Arabic numerals and referred to in the text as e.g. Fig. 2 or Figs. 2–5. Photographs should be given only if essential and should be enlarged enough to permit clear reproduction.

The Places of tables and figures within the text have to be marked with lead pencil on the left margin of the manuscript.

References are indicated in the text by the author's name and year of publication. They should be listed in alphabetic and chronologic order at the end of the paper, as follows: name and the initials of the author(s), the year of publication, suitable abbreviation of the journal (or title of the book and editing house), its volume and page.

The content of the papers will be introduced on floppy disk (3.5"), in well-known editor, preferably Word v. 6. The collections and the emphasis will be made respecting (as far as possible) the prototype of the journal, 11/13 font for the text proper, 12/14 for the paper titles and 9/11 for annexes (tables, references, explanation, footnotes, etc.).

Please pay attention to these recommendations; it will contribute to a faster publishing of the manuscript.

ROMANIAN ASTRONOMICAL JOURNAL

Vol. 7, No. 2, 1997

CONTENTS

ISSN 1210-5168

Rom. Astron. J., Vol. 7, No. 2, p. 119–208, Bucharest, 1997

Lei 3 200