ROMANIAN ACADEMY

ROMANIAN ASTRONOMICAL JOURNAL

Vol. 8, No. 1
1998

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ROMANIAN ACADEMY

ROMANIAN ASTRONOMICAL JOURNAL

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ROMANIAN ASTRONOMICAL JOURNAL

Vol. 8, No. 1, 1998

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Rom. Astron. J., Vol. 8, No. 1, p. 1–58, Bucharest, 1998

IMMERSION SURFACE FOR A NUMERICAL MODEL OF NEUTRON STAR

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Abstract. The spacetime geometry is being presented for a numerical model of neutron star whose equation of state given tabularly and fitted by cubic splines. Three-dimensional immersion diagram for the relativistic model are traced.

Key words: astrophysics - stellar structure - neutron stars - 3Dimmersion diagram

1. MATHEMATICAL MODEL

The mathematical model which describes the structure of a neutron star is given by Tolman-Oppenheimer-Volkoff (TOV) equations (see Tolman 1939):

$$\frac{\mathrm{d}M(r)}{\mathrm{d}r} = 4\pi r^2 \rho(r), \tag{1}$$

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G\left[\rho(r) + \frac{P(r)}{c^2}\right]\left[M(r) + 4\pi r^3 \frac{P(r)}{c^2}\right]}{r^2\left(1 - \frac{2GM(r)}{rc^2}\right)}, \tag{2}$$

$$P = P(\rho), \tag{3}$$

where $P(r)$ = pressure, $\rho(r)$ = total mass density, $M(r)$ = gravitational mass (all inside the spherical domain of radius r), G = Newtonian gravitational constant, and c = speed of light. Using the transformations (Ureche 1980):

$$r = a\eta\,,\ \rho = \rho_c \Psi,\ \ P = \rho_c c^2 p,\ \ M(r) = M^* m\,, \tag{4}$$

where ρ_c = central density, a and M^* = scale factors (of dimension length and mass, respectively), the system (1)-(3) acquires the form:

$$\frac{dm}{d\eta} = \eta^2 \psi , \tag{5}$$

$$\frac{dp}{d\eta} = -\frac{(\psi + p)(m + \eta^3 p)}{\eta^2 \left(1 - \frac{2m}{\eta}\right)}, \tag{6}$$

$$p = p(\psi) , \tag{7}$$

with the boundary conditions:

$$m(0) = 0 ,\ \Psi(\eta_s) = 0,\ \ p(\eta_s) = 0, \tag{8}$$

where $\eta_s = R/a$ is the value of the non-dimensional radial coordinate η at the surface of the star.

To integrate numerically the system (6)-(7), the equation of state, given tabularly, was fitted by cubic splines (Pop *et al.* 1995).

2. SPACETIME GEOMETRY

In dimensionless form, the spacetime geometry inside and in the neighbourhood of the neutron star is described respectively by

$$\frac{1}{2}(\psi + p)\frac{d\nu}{d\eta} + \frac{dp}{d\eta} = 0 ,$$

$$\lim_{\eta \to \eta_s} \nu(\eta) = \nu(\eta_s) \equiv \nu_s , \tag{9}$$

$$\nu(0) = \nu_c ,$$

for $\boldsymbol{\eta} < \boldsymbol{\eta}_s$, and

$$e^\nu = 1 - \frac{2m_s}{\eta}, \tag{10}$$

for $\boldsymbol{\eta} \geq \boldsymbol{\eta}_s$,

$$e^{-\lambda} = 1 - \frac{2m}{\eta}, \text{ for } \eta < \eta_s \tag{11}$$

and

$$e^{-\lambda} = 1 - \frac{2m_s}{\eta}, \text{ for } \eta \geq \eta_s, \tag{12}$$

in which $m_s = m(\eta_s)$ is the value of the dimensionless mass at the star surface.

The function $\nu = \nu(\eta)$ appears in the relationship between the proper (local) time τ and the observer's time t. As to $\lambda = \lambda(\eta)$, it relates the (local) radial distance to the radial coordinate (Zeldovich, Novikov, 1971).

The values of $\nu = \nu(\eta)$ are obtained by integrating numerically the system:

$$\frac{dm}{d\eta} = \Psi \eta^2,$$

$$\frac{dp}{d\eta} = -\frac{(\Psi + p)(m + p\eta^3)}{\eta^2\left(1 - \frac{2m}{\eta}\right)}, \tag{13}$$

$$\frac{d\nu}{d\eta} = -\frac{2}{\Psi + p}\frac{dp}{d\eta},$$

with the initial conditions:

$$\nu(\eta_s) = 0.4094,$$
$$m(\eta_s) = 1.129571,$$
$$p(\eta_s) = 0,$$

provided by the considered model. The results are listed in Table 1.

Tabel 1

The distribution of $m(\eta)$, $\nu(\eta)$ and $e^{-\nu/2}$ from surface towards the centre

η	$m(\eta)$	$\nu(\eta)$	$e^{-\nu/2}$
3.81700000	1.12358513	.37639876	.82844951
3.61700000	1.09738601	.29525720	.86275148
3.41700000	1.05932737	.20180586	.90402078
3.21700000	1.01255173	.09479104	.95371011
3.01700000	.95870102	-.02728598	1.01373648
2.81700000	.89881647	-.16610484	1.08659877
2.61700000	.83362089	-.32346558	1.17554608
2.41700000	.76364791	-.50119282	1.28479145

Table 1 (continued)

η	$m(\eta)$	$\nu(\eta)$	$e^{-\nu/2}$
2.21700000	.68932449	-.70097300	1.41975809
2.01700000	.61104920	-.92408655	1.58731400
1.81700000	.52929493	-1.17099028	1.79587998
1.61700000	.44476711	-1.44070779	2.05516039
1.41700000	.35865247	-1.73001360	2.37502224
1.21700000	.27298327	-2.03247400	2.76277884
1.01700000	.19107770	-2.33755521	3.21805649
.81700000	.11783163	-2.63021229	3.72514637
.61700000	.05929926	-2.89151517	4.24506692
.41700000	.02076554	-3.10072984	4.71318979
.21700000	.00318328	-3.23870009	5.04980711
.01700000	.00000158	-3.29162048	5.18520946

3. IMMERSION DIAGRAM

To obtain an intuitive picture of the spacetime behaviour in the neighbourhood of our model, let V_3 be the curved Riemannian 3-space endowed with the metric:

$$ds^2 = e^{\lambda}dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\varphi^2. \tag{15}$$

Consider an equatorial section ($\theta = \pi/2$) in this space; we obtain a $2D$ manifold $V_2 \subset V_3$, with the metric:

$$ds_2^2 = e^{\lambda}\,dr^2 + r^2d\varphi^2, \tag{16}$$

Also write the Euclidean metric E_3 in cylindrical coordinates:

$$ds_3^2 = dr^2 + r^2d\varphi^2 + dZ^2. \tag{17}$$

Considering V_2 immersed into E_3, we have (Misner *et al.* 1973):

$$ds_3^2 = \left[1 + \left(\frac{dZ}{dr}\right)^2\right]dr^2 + r^2d\varphi^2\,. \tag{18}$$

Identifying (16) and (18), we get the equation:

$$1 + \left(\frac{dZ}{dr}\right)^2 = e^{\lambda}\,, \tag{19}$$

which is integrated with the initial condition $Z(0) = 0$.

The surface $Z = Z(r)$ depicts the geometric properties of the space in the neighbourhood of the relativistic star. Taking into account the relativistic equation of hydrostatic equilibrium, one obtains the immersion function:

$$Z(r)=\int_0^r\sqrt{\frac{2GM(r)}{c^2r-2GM(r)}}\mathrm{d}r\ ,\qquad r\in[0,+\infty). \tag{20}$$

Since $M(\mathrm{R}) = M$ = constant for $r \geq R$ (outside the star), we may write:

$$Z(r)=\int_0^R\sqrt{\frac{2GM(r)}{c^2r-2GM(r)}}\mathrm{d}r+\int_R^r\sqrt{\frac{2GM}{c^2r-2GM}}\mathrm{d}r, \tag{21}$$

or

$$Z(r)=\int_R^r\sqrt{\frac{2GM}{c^2r-2GM}}\mathrm{d}r+Z(R), \tag{22}$$

where $Z(R) = Z(r = R)$.

The integral (22) reads:

$$\int_R^r\sqrt{\frac{2GM}{c^2r-2GM}}\mathrm{d}r=2\sqrt{R_g}\left(\sqrt{r-R_g}-\sqrt{R-R_g}\right), \tag{23}$$

where $R_g = 2GM/c^2$. By virtue of (20)-(23), the final expression of $Z = Z(r)$ becomes:

$$Z(r)=\begin{cases}\displaystyle\int_0^r\sqrt{\frac{2GM(r)}{c^2r-2GM(r)}}\mathrm{d}r, & \text{for } 0\leq r\leq R,\\ 2\sqrt{R_g}\left(\sqrt{r-R_g}-\sqrt{R-R_g}\right)+Z(R), & \text{for } r\geq R,\end{cases} \tag{24}$$

the function Z being continuous for $r = R$.

Using (4) and

$$Z = az. \tag{25}$$

the expression (24) acquires the dimensionless form:

$$z(\eta)=\begin{cases}\displaystyle\int_0^\eta\sqrt{\frac{2m}{\eta-2m}}\mathrm{d}\eta & \text{for } 0\leq\eta\leq\eta_s,\\ 2\sqrt{m_s}\left(\sqrt{\eta-2m_s}-\sqrt{\eta_s-2m_s}\right)+z(\eta_s), & \text{for } \eta\geq\eta_s.\end{cases} \tag{26}$$

Obviously, the surface $z = z(\eta)$ also features the geometric properties of the space in the neighbourhood of the star. The curves which represent the sections of $z = z(\eta)$ with the planes φ =constant, calculated for our model, are plotted in Figure 1.

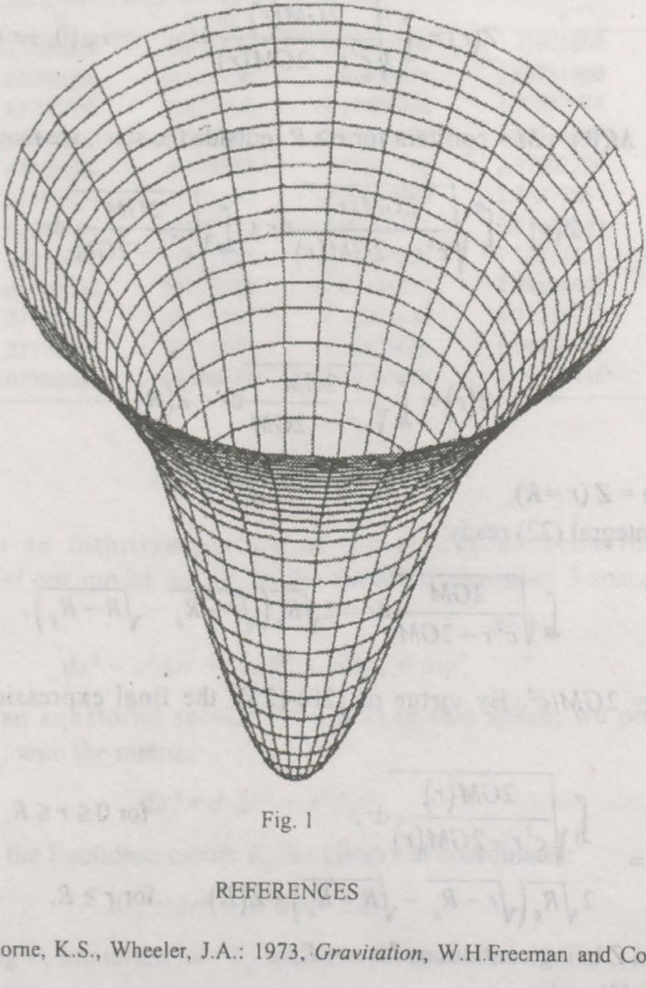

Fig. 1

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Received on 15 December 1997

STELLAR PERIOD VARIABILITY: THE EQUIVALENCE BETWEEN POLYNOMIAL AND MULTIPERIODIC EPHEMERIDES

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Abstract. The present approach deals with the problem of establishing a relationship between two essentially different types of ephemerides for period variability phenomena – a high degree polynomial one, and a superposition between a low degree polynomial and a multiperiodic ephemeris. Such ephemerides are able to describe the observational manifestations caused by the same physical mechanisms of a given variable star. The final goal we have in view in this preliminary approach is to sketch a way for finding the best model for the period variability of a variable star, having also the maximum physical relevance.

Key words: variable stars, data analysis

The study of period variability phenomena in variable stars – especially in eclipsing close binary systems (*e.g.*, Algols, contact binaries, or RS CVn binaries) – on the basis of large observational data sets (extremum light times covering sometimes a century) often reveals intricate situations. The observed behaviour emphasized by the specific shape of the corresponding *O–C* curves could be understood having in view different physical mechanisms possibly occurring (maybe simultaneously) in these stars (see, *e.g.*, the excellent reviews in the papers of Hall (1990) and Kalimeris (1994a) as well as the papers of Mayer (1988, 1990), Fernie (1990); see also Pop (1996), Pop *et al.* (1996) and references therein). There are such *O–C* curves that often could be described by high degree polynomial ephemerides (see, *e.g.*, Wood & Forbes (1963), Fernie (1990), Demircan & Derman (1992), Derman & Demircan (1992), Kalimeris *et al.* (1994a, 1994b, 1995), Rovithis-Livaniou, *et al.*, 1994). On the other hand, there are situations in which a superposition of a parabolic or cubic ephemeris (or, two or more linear ephemerides corresponding to consecutive time intervals) and a periodic term (*e.g.*, Todoran, 1980; Rafert, 1982, Applegate, 1992) is able to fit well the *O–C* curve. Finally, we mention the case of AR Lac; in order to explain the complex features appearing in this situation, Kim (1991) proposed a beat phenomenon, *i.e.* a multiperiodic modulation.

Taking into account the above mentioned aspects, we found that frequently, such complicate *O–C* curves can be described very accurately using an ephemeris with a high degree of generality involving the superposition of a *polynomial term* and a *multiperiodic term* (see Pop 1996, Pop *et al.* 1996)

$$t_{n_j} = t_0 + P_p n_j + \sum_{q=2}^{Q} \tau_q^p n_j^q + \sum_{l=1}^{L} \sum_{m=1}^{M_l} \tau_{lm} \sin\left(\Omega_{lm} n_j + \varphi_{lm}\right), \tag{1}$$

in which

$$\begin{aligned} \Omega_{lm} &= m\Omega_l, \\ \Omega_l &= 2\pi f_{0l} = 2\pi\left(P_p^0 / P_{sl}\right), \end{aligned} \tag{2}$$

and $Q \leq 3$. At the same time, a high degree polynomial ephemeris (a "*pure*" *polynomial ephemeris*) fits equally well the considered *O–C* curve

$$t_{n_j} = t_0 + P_p n_j + \sum_{k=2}^{K} \tau_k n_j^k, \tag{3}$$

with, *e.g.*, $K \leq 11$. Note that such a situation is especially critical when we are disposing only of two or three cycles of the modulating signal(s) in the *O–C* diagram; because of the too short available base time, we are not able to detect possible low frequency (*i.e.* long periodic) spectral components. The aim of our approach is to establish a relationship between the parameters of the ephemerides defined by Eqs (1) and (3), in an attempt to achieve additional information on the observed period variability phenomenon.

Now, let us consider a variable star period variability phenomenon described by the following multiperiodic ephemeris

$$t_{n_j} = t_0 + P_p n_j + \sum_{l=1}^{L} \sum_{m=1}^{M_l} \tau_{lm} \sin\left(\Omega_{lm} n_j + \varphi_{lm}\right). \tag{4}$$

One can rewrite it in the form

$$t_{n_j} = t_0 + P_p n_j + \sum_{l=1}^{L} \sum_{m=1}^{M_l} a_{lm} \sin\left(\Omega_{lm} n_j\right) + \sum_{l=1}^{L} \sum_{m=1}^{M_l} b_{lm} \cos\left(\Omega_{lm} n_j\right), \tag{5}$$

where

$$\begin{aligned} a_{lm} &= \tau_{lm} \cos\varphi_{lm}, \\ b_{lm} &= \tau_{lm} \sin\varphi_{lm}. \end{aligned} \tag{6}$$

Using the well known expansions for the functions sine and cosine (*e.g.*, Rîjic & Gradştein, 1955)

$$\sin x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!},$$
$$\cos x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}, \tag{7}$$

the multiperiodic ephemeris (5) becomes a polynomial type one

$$t_{n_j} = t_0^l + P_p^l n_j + \sum_{k=1}^{K}\left(\tau_{2k} n_j^{2k} + \tau_{2k+1} n_j^{2k+1}\right), \tag{8}$$

which we call hereafter *equivalent polynomial ephemeris*. Here we used the following notations

$$t_0^l = t_0 + \sum_{l=1}^{L}\sum_{m=1}^{M_l} b_{lm},$$
$$P_p^l = P_p + \sum_{l=1}^{L}\sum_{m=1}^{M_l} a_{lm}\Omega_{lm}, \tag{9}$$

and

$$\tau_{2k} = \frac{(-1)^k}{(2k)!}\sum_{l=1}^{L}\sum_{m=1}^{M_l} b_{lm}\Omega_{lm}^{2k},$$
$$\tau_{2k+1} = \frac{(-1)^k}{(2k+1)!}\sum_{l=1}^{L}\sum_{m=1}^{M_l} a_{lm}\Omega_{lm}^{2k+1}, \tag{10}$$

and we truncated the series to the K^{th} order.

Thus, we have established the equivalence between the multiperiodic ephemeris (Eq. (5)) and the polynomial ephemeris (Eq. (8)). With this result in mind, we shall return to the concrete situation considered at the beginning of this paper (see Eqs (1) and (3)). Here we shall assume that all the parameters appearing in these formulae – determined through linear or nonlinear least squares fitting – are statistically significant. We also shall presume that we have obtained several sets of values for the parameters in ephemeris (1), corresponding to different values for Q, L, and M_l (l=1,2, ..., L). Under such conditions we can compare the values of the coefficients of the *"pure" polynomial ephemeris* (3) with those of the *equivalent polynomial ephemeris* (8) using the following formulae, which are derived immediately

$$\tau_{2k} = \tau_{2k}^{p} + \frac{(-1)^k}{(2k)!} \sum_{l=1}^{L} \sum_{m=1}^{M_l} b_{lm} \Omega_{lm}^{2k},$$
$$\tau_{2k+1} = \tau_{2k+1}^{p} + \frac{(-1)^k}{(2k+1)!} \sum_{l=1}^{L} \sum_{m=1}^{M_l} a_{lm} \Omega_{lm}^{2k+1}, \tag{11}$$

for terms with $Q \leq K$, together with Eqs (9).

As it can easily be seen, the application of the above derived formulae is straightforward enough, so that numerical tests (using various combinations for Q, L, and M_p, $l = 1,2, ..., L$) can be performed without any difficulty. The ultimate goal of such numerical tests is that of obtaining additional information concerning the optimum degree for the polynomial describing the secular trend in the O–C diagram. At the same time we hope that testing such an equivalence between the two types of models for period variability could provide an additional way for a better understanding of the behaviour of the O–C curve.

Acknowledgements. It is a pleasure for the author to express his gratitude to Dr. Ioan Todoran for illuminating discussions leading to this paper. Thanks are also due to Professors Douglas Hall, Pavel Mayer and Helen Rovithis-Livaniou for kindly putting at our disposal reprints of their papers.

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Received on 12 April 1998

TWO NEW DYNAMICAL MODELS IN CELESTIAL MECHANICS

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Abstract. The Newtonian restricted (n+1)-body problem is being approached. Taking successively as basis a barycentric frame and a relative one, the barycentric restricted (n+1)-body problem and the relative restricted (n+1)-body problem are defined. On the basis of the symmetrical solutions of the n-body problems defined by the n finite masses, the barycentric and relative circular restricted problems are pointed out. The existence of the Jacobi integral for both problems is established. The expression of the two Jacobi integrals (one for each problem) is the same, but the rotation parameters differ one another.

Key words: celestial mechanics, restricted many-body problem, Jacobi integral

1. BRIEF HISTORY OF THE PROBLEM

One of the best known models in astronomy, the *restricted three-body problem,* is still keeping its outstanding theoretical and practical importance. Poincaré`s (1892, 1894) famous results imposed this name, which is generally accepted today.

Recall the statement of this problem: *Study all possible motions (in the 3-dimensional configuration space) of a material point P of negligible mass in the Newtonian gravitational field generated by the attraction of two material points P_0 and P_1 of finite and nonzero masses m_0 and m_1.* The bodies P_0 and P_1 move in space under the only action of their reciprocal Newtonian attraction, hence their motion is fully determined by the differential equations of the two-body problem, whose general solution is well-known. Geometrically, this solution is represented by a conic section (ellipse, parabola, hyperbola, or the limit cases of these ones).

The differential equations that describe the motion of the point P form a nonlinear and nonautonomous system of sixth order which can be written as (e.g. Abalakin et al.1976):

$$\begin{aligned} \ddot{\xi} &= \partial U_2(\xi,\eta,\zeta,t) / \partial\xi, \\ \ddot{\eta} &= \partial U_2(\xi,\eta,\zeta,t) / \partial\eta, \\ \ddot{\zeta} &= \partial U_2(\xi,\eta,\zeta,t) / \partial\zeta, \end{aligned} \tag{1}$$

where (ξ,η,ζ), are the barycentric coordinates of P in the barycentric frame $G\xi\eta\zeta$, G is the barycentre of the system (P_0, P_1), the dots mark time-differentiation, while the force function of the problem (U_2) reads

$$U_2(\xi,\eta,\zeta,t)=f\left(m_0/\Delta_0+m_1/\Delta_1\right), \tag{2}$$

In (2), f stands for the Newtonian gravitational constant, while Δ_0 and Δ_1 have the expressions

$$\begin{aligned}\Delta_0^2&=(\xi-\xi_0(t))^2+(\eta-\eta_0(t))^2+(\zeta-\zeta_0(t))^2,\\ \Delta_1&=(\xi-\xi_1(t))^2+(\eta-\eta_1(t))^2+(\zeta-\zeta_1(t))^2,\end{aligned} \tag{3}$$

$\xi_k(t)$, $\eta_k(t)$, $\zeta_k(t)$ being the barycentric coordinates of the body P_k, $k=0,1$ (known functions of the time t).

If the motion of P_0 and P_1 is performed on circular orbits with respect to the barycentre G, with the angular velocity ω_2 , then it is convenient to use a uniformly rotating barycentric frame Gxyz (whose plane Gxy coincides with $G\xi\eta$), in which equations (1) become autonomous :

$$\begin{aligned}\ddot{x}-2\omega_2\dot{y}&=\partial V_2(x,y,z)/\partial x,\\ \ddot{y}+2\omega_2\dot{x}&=\partial V_2(x,y,z)/\partial y,\\ \ddot{z}&=\partial V_2(x,y,z)/\partial z,\end{aligned} \tag{4}$$

where

$$V_2(x,y,z)=\left(\omega_2^2/2\right)\left(x^2+y^2\right)+U_2^*(x,y,z), \tag{5}$$

$$U_2^*(x,y,z)\equiv U_2(\xi,\eta,\zeta,t). \tag{6}$$

Jacobi (1884) and Poincaré (1892, 1894) showed that equations (4) admit the first integral

$$\dot{x}^2+\dot{y}^2+\dot{z}^2=2V_2(x,y,z)+2h, \tag{7}$$

called today *Jacobi integral* (at Poincaré's proposal).

In all other cases (of conic section), the introduction of the uniformly rotating frame does not remove the time t from the right-hand sides of equations (1), therefore the equations of the form (4) do not become autonomous. In this situation one cannot establish a first integral analogous to the Jacobi integral. We also have to mention that many studies have been dedicated to the different senses of stability of the equilibrium solutions (equilibrium positions) of equations (4). Among them we quote those by Wintner (1941), Kolmogorov (1961), Arnold (1974), Markeev (1978).

2. THE RESTRICTED (*n*+1)-BODY PROBLEM

We call *Newtonian restricted* $(n+1)$*-body problem* the following dynamical model: study (in the 3-dimensional configuration space) the motion of the material point P of negligible mass in the Newtonian gravitational field generated by the attraction of n material points P_k, of finite nonzero masses $m_k\,(k=\overline{0,n-1})$, which move in a barycentric frame, or in a relative one, on orbits determined by the equations of the two-body problem.

Such a model does not violate the laws of the classical dynamics, because it was proved (Wintner 1941; Elmabsout 1988; Grebenicov 1997) that, for certain initial conditions, the $(n+1)$-body problem with finite nonzero masses is Liouvillian (Wintner 1941). Moreover, it was proved that there exists a nonempty set of initial data, which form a 3-dimensional manifold $\{\mathcal{M}_3\}$ in the phase space of coordinates and velocities, and which generates homographic barycentric solutions of the n-body problem (in Wintner's sense; see Wintner 1941), or relative solutions, symmetrical with respect to the centre P_0 (Elmabsout 1988; Grebenicov 1997). If the initial positions of the bodies P_k $(\xi_k(0),\ \eta_k(0),\ \zeta_k(0))$ belong to $\{\mathcal{M}_3\}$ for $(k=\overline{0,n-1})$, then the regular polygon (with n sides) $P_0P_1\ldots P_{n-1}$ will rotate around the barycentre G of the system, and every vertex P_k will move on a Keplerian orbit, namely the coordinates

$$\begin{aligned}\chi_k(t) &= \chi_k\big(t,\chi_k(0)\big),\\ \chi_k(0) &= \chi_k\big(0,\chi_k(0)\big),\end{aligned}\tag{8}$$

(with $\chi\in\{\xi,\ \eta,\ \zeta\}$ and $(k=\overline{0,n-1})$ are known functions of t.

For the symmetrical solutions (Elmabsout 1988; Grebenicov 1997), as functions of t, one considers the coordinates $u_k(t)$, $v_k(t)$, $w_k(t)$ of the vertices P_k $(k=\overline{0,n-1})$ of the $(n-1)$-sided polygon $P_1P_2\ldots P_k$ in the relative Cartesian frame P_0uvw.

This qualitative analysis allows us to propose two dynamical models for the Newtonian restricted $(n+1)$-body problem; we shall call them : *the barycentric restricted* $(n+1)$*-body problem* and *the relative restricted* $(n+1)$*-body problem.*

Within the framework of the barycentric restricted $(n+1)$-body problem, the motion equations of P have the form (1), but, instead of U_2 , one must use the force function

$$U_n(\xi,\mu,\zeta,t)=f\sum_{k=0}^{n-1}m_k/\Delta_k,\tag{9}$$

where

$$\Delta_k^2=\big(\xi-\xi_k(t)\big)^2+\big(\eta-\eta_k(t)\big)^2+\big(\zeta-\zeta_k(t)\big)^2.\tag{10}$$

and the barycentric coordinates of the body P_k, $(\xi_k, \eta_k, \zeta_k)(t)$, are determined from Wintner's homographic solutions.

Within the framework of the second model, that of the relative restricted (n+1)-body problem, the motion equations of P have the form (see e.g. Abalakin et al. 1976):

$$\ddot{u} + fm_0 u/r^3 = \partial R_{n-1}(u,v,w,t)/\partial u\,,$$

$$\ddot{v} + fm_0 v/r^3 = \partial R_{n-1}(u,v,w,t)/\partial v, \qquad (11)$$

$$\ddot{w} + fm_0 w/r^3 = \partial R_{n-1}(u,v,w,t)/\partial w,$$

where the perturbing function $R_{n-1}(u,v,w,t)$ is expressed via the well-known relations (e.g. Abalakin et al. 1976) :

$$R_{n-1}(u,v,w,t) = fm\sum_{k=1}^{n-1}\left[1/\Delta_k - (uu_k + vv_k + ww_k)/r_k^3\right],$$

$$r^2 = u^2 + v^2 + w^2,\, r_k^2 = u_k^2 + v_k^2 + w_k^2\,, \qquad (12)$$

$$\Delta_k^2 = (u-u_k)^2 + (v-v_k)^2 + (w-w_k)^2;$$

with $k=\overline{1,n-1}$. Here $(u_k(t), v_k(t), w_k(t))$ and (u,v,w) are respectively the Cartesian relative coordinates of the points P_k ($k=\overline{1,n-1}$) and P is the P_0uvw frame.

Elmabsout (1988) and Grebenicov (1997) proved that the second model can be realized only when the masses m_k of the bodies P_k ($k=\overline{1,n-1}$) are all equal. This condition entails the apparition of the factor m ($= m_k$, $k=\overline{1,n-1}$) is front of the sum in the first formula (12).

3. JACOBI INTEGRALS

Consider first the barycentric restricted (n+1)-body problem. If the homographic solution (in Wintner's sense) of the n-body problem is such that the bodies $P_0, P_1 \ldots P_{n-1}$ move on circular orbits around the barycentre G with the constant angular velocity ω_n^*, i.e. the bodies form a uniformly rotating n-sided polygon of constant size, then one can pass to the uniformly rotating frame $Gxyz$. In these new coordinates, instead of (1) endowed with the function (9), we obtain the *autonomous* system

$$\ddot{x} - 2\omega_n^* \dot{y} = \partial V_n(x,y,z)/\partial x,$$
$$\ddot{y} + 2\omega_n^* \dot{y} = \partial V_n(x,y,z)/\partial y, \tag{13}$$
$$\ddot{z} = \partial V_n(x,y,z)/\partial z,$$

where

$$V_n(x,y,z) = \left(\omega_n^{*2}/2\right)\left(x^2+y^2\right) + U_n^*(x,y,z), \tag{14}$$

$$U_n^*(x,y,z) \equiv U_n(\xi,\eta,\zeta,t), \tag{15}$$

Using Poincaré's (1892, 1894) method, we easily find a first integral of (13) – *the Jacobi integral of the Newtonian barycentric circular restricted* $(n+1)$*-body problem* :

$$\left(\ddot{x}+\ddot{y}+\ddot{z}\right) = 2V_n(x,y,z) + 2h, \tag{16}$$

where h is an arbitrary constant of integration.

In the relative circular restricted $(n+1)$-body problem, the situation is analogous. If we introduce the relative frame P_0XYZ which rotates uniformly with the angular velocity ω_{n-1} (Grebenicov 1997) :

$$\omega_{n-1}^2 = f\left[m_0 + \frac{m}{4}\sum_{k=2}^{n-1}\left(\sin\frac{\pi(k-1)}{n-1}\right)^{-1}\right]\bigg/ a_0^3, \tag{17}$$

where a_0 $(=P_0P_k,\ k=\overline{1,n-1})$ is the radius of the circle circumscribed to the polygon, then equations (11) turn to the *autonomous* system

$$\ddot{X} - 2\omega_{n-1}\dot{Y} = \omega_{n-1}^2 X - fm_0 X/r^3 + \partial R_{n-1}^*(X,Y,Z)/\partial X,$$
$$\ddot{Y} - 2\omega_{n-1}\dot{X} = \omega_{n-1}^2 Y - fm_0 Y/r^3 + \partial R_{n-1}^*(X,Y,Z)/\partial Y, \tag{18}$$
$$\ddot{Z} = -fm_0 Z/r^3 + \partial R_{n-1}^*(X,Y,Z)/\partial Z,$$

where the perturbing function R_{n-1}^* depends on the new rotating coordinates (X,Y,Z). It is expressed by the formula

$$R_{n-1}(u,v,w,t) \equiv R_{n-1}^*(X,Y,Z) = fm\sum_{k=1}^{n-1}\left[1/\Delta_k - \left(XX_k + YY_k + ZZ_k\right)/r_k^3\right], \tag{19}$$

where $r_k = a_0$, $X_k = a_0\cos\frac{2\pi(k-1)}{n-1}$, $Y_k = a_0\sin\frac{2\pi(k-1)}{n-1}$, $Z_k = 0$, $k=\overline{1,n-1}$.

Multiplying equations (18) respectively by $2\dot{X}, 2\dot{Y}, 2\dot{Z}$, and adding together the resulting expressions, we get the differential equality

$$\mathrm{d}\left(\dot{X}^2+\dot{Y}^2+\dot{Z}^2\right)/\mathrm{d}t=\omega_{n-1}^2\mathrm{d}\left(2X\dot{X}+2Y\dot{Y}\right)/\mathrm{d}t+2fm_0\mathrm{d}(1/r)/\mathrm{d}t+2\mathrm{d}R_{n-1}^*+h, \quad (20)$$

which, after integration, provides the Jacobi integral for the second model of the restricted circular $(n+1)$-body problem

$$\left(\dot{X}^2+\dot{Y}^2+\dot{Z}^2\right)/2=\left(\omega_{n-1}^2/2\right)\left(X^2+Y^2\right)+fm_0/r+R_{n-1}^*+h, \quad (21)$$

where $r^2=X^2+Y^2+Z^2$, and h is an arbitrary constant of integration.

Introducing the new function W_{n-1} via

$$W_{n-1}(X,Y,Z)=fm_0/r+fm\sum_{k=1}^{n-1}\left[1/\Delta_k-\left(XX_k+YY_k\right)/r_k^3\right], \quad (22)$$

the *Jacobi integral of the relative circular restricted* $(n+1)$*-body problem* can be expressed as

$$\left(\dot{X}^2+\dot{Y}^2+\dot{Z}^2\right)/2=\left(\omega_{n-1}^2/2\right)\left(X^2+Y^2\right)+W_{n-1}(X,Y,Z)+h, \quad (23)$$

Now we take into account the necessary conditions for the existence of the symmetrical solution of the (n+1)-body problem with respect to the axes P_0X and P_0Y (Grebenicov 1997)

$$\sum_{k=1}^{n-1}X_k=0=\sum_{k=1}^{n-1}Y_k,$$

and introduce the notation $r=\Delta_0$. Then the function W_{n-1} coincides with the force function (15), so the final form of the Jacobi integral for the second model of the circular restricted $(n+1)$-body problem coincides with the barycentric form (16), although the rotation parameters ω_n^* and ω_{n-1} differ each other. In more detail,

$$\left(\dot{X}^2+\dot{Y}^2+\dot{Z}^2\right)/2=\left(\omega_{n-1}^2/2\right)\left(X^2+Y^2\right)+fm_0/\Delta_0+fm\sum_{k=1}^{n-1}\left(1/\Delta_k\right)+h, \quad (24)$$

To end, let us point out the mechanical meaning of the Jacobi integral (24). One sees that the relative total energy of the particle P (the energy in the P_0XYZ *frame*), consisting of the sum of the reduced kinetic energy, the centrifugal energy, and the potential energy of the attracting masses, is constant and equal to h.

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Received on 20 December 1997

JACOBI INTEGRAL IN THE RESTRICTED CIRCULAR (n+1)-BODY PROBLEM WITH HOMOGENEOUS POTENTIAL

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Abstract. The existence of the Jacobi integral for the restricted circular (n+1)-body problem ($n \geq 3$) is proved. We consider n points, P_0, P_1, ... P_{n-1}, of masses m_0, m_1 = ... = m_{n-1} = m, respectively, in a field featured by central forces $F_{ks} \sim \Delta_{ks}^{-(\alpha+1)}$ (for $\alpha = 1$ we have the Newtonian law). For some symmetric initial conditions, there are solutions where bodies P_1, ... P_{n-1} are rotating around P_0 with constant angular velocities. The points P_1, ..., P_{n-1} lie in a fixed plane, and form a regular (n-1)-polygon centered in P_0. In such a case, for the system of differential equations which describe the three-dimensional motion of a massless point P in the field generated by the bodies P_0, P_1, ... , P_{n-1}, one can find the Jacobi integral.

Key words: celestial mechanics - homogeneous potentials - (n+1)-body problem - Jacobi integral.

Grebenicov (1998) proved the existence of the Jacobi integral for two dynamical models characterizing special cases of the Newtonian (n+1)-body problem, with $n \geq 3$, in which the forces of reciprocal attraction between the bodies P_k and P_s are of the type $F_{ks} \sim \Delta_{ks}^{-2}$, Δ_{ks} = distance between P_k and P_s.

In the present paper we prove the existence of the Jacobi integral in the restricted circular (n+1)-body problem, $n \geq 3$, for a more general type of attraction. Within this dynamical model, the bodies P_0, P_1, ... P_{n-1}, of masses m_0, m_1 = ... = m_{n-1} = m attract both one another and the infinitesimal body P (of negligible mass) with forces of the type $F_{ks} \sim \Delta_{ks}^{-(\alpha+1)}$, where Δ_{ks} is the distance between the points P_k and P_s, whereas $\alpha > 0$ is an arbitrary parameter. Gadomski et al. (1997) proved that in such a model, for certain initial conditions, the points P_1, ... , P_{n-1} move around P_0 in a fixed plane, on circular orbits, forming a regular (n-1)-polygon all along the motion.

The differential equations which describe the motion of the point P of negligible mass in the relative Cartesian frame (P_0xyz), for which the (P_0xy)-plane coincides with the plane to which the motion of the points P_0, P_1, ... P_{n-1} is confined, read (e.g. Abalakin et al. 1976):

$$\ddot{x} + \frac{fm_0 x}{r^{\alpha+2}} = \frac{\partial R_{n-1}}{\partial x},$$
$$\ddot{y} + \frac{fm_0 y}{r^{\alpha+2}} = \frac{\partial R_{n-1}}{\partial y}, \qquad (1)$$
$$\ddot{z} + \frac{fm_0 z}{r^{\alpha+2}} = \frac{\partial R_{n-1}}{\partial z},$$

where

$$R_{n-1}(x,y,z,t) = fm\sum_{k=1}^{n-1}\left(\frac{1}{\Delta_k^{\alpha}} - \frac{xx_k + yy_k + zz_k}{r_k^{\alpha+2}}\right),\ r^2 = x^2 + y^2 + z^2,\ z_k = 0, \qquad (2)$$
$$\Delta_k^2 = (x - x_k)^2 + (y - y_k)^2 + z^2,\quad r_k^2 = x_k^2 + y_k^2, \qquad (3)$$

f – Newtonian gravitational constant, (x,y,z) – coordinates of the point P, $(x_k, y_k, 0)$ – coordinates of the point P_k, while the notations $\ddot{x}, \ddot{y}, \ddot{z}$ signify time-derivatives, for instance $\ddot{x} = \ddot{x}(t) = \dfrac{d^2 x}{dt}$.

In equations (1) the coordinates (x, y, z) are unknown functions of t, while the coordinates (x_k, y_k) are known functions of t, that is why R_{n-1} depends explicitly on t.

Using Poincaré's (1965) method, we pass to the rotating frame (P_0XYZ) via

$$x = X\cos v(t) - Y\sin v(t),$$
$$y = X\sin v(t) + Y\cos v(t), \qquad (4)$$
$$z = Z,$$

where the variable angle $v(t)$ is, for the moment, an arbitrary differentiable function of the time t.

In the new variables, equations (1) acquire the form

$$\ddot{X} - 2\dot{Y}\dot{v} - X\dot{v}^2 - Y\ddot{v} = -\frac{fm_0\alpha X}{r^{\alpha+2}} + \frac{\partial R^*_{n-1}}{\partial X},$$
$$\ddot{Y} + 2\dot{X}\dot{v} - Y\dot{v}^2 + X\ddot{v} = -\frac{fm_0\alpha Y}{r^{\alpha+2}} + \frac{\partial R^*_{n-1}}{\partial Y}, \qquad (5)$$
$$\ddot{Z} = -\frac{fm_0\alpha Z}{r^{\alpha+2}} + \frac{\partial R^*_{n-1}}{\partial Z},$$

$$R^*_{n-1}(X,Y,Z,t) \equiv R_{n-1}(x,y,z,t),\quad r^2 = X^2 + Y^2 + Z^2 \qquad (6)$$

Of course, $\Delta_k^2 = (X - X_k)^2 + (Y - Y_k)^2 + Z^2$.

Multiplying equations (5) by $2\dot{X}, 2\dot{Y}, 2\dot{Z}$, respectively, then adding them together, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{X}^2+\dot{Y}^2+\dot{Z}^2\right)-\dot{\nu}\frac{\mathrm{d}}{\mathrm{d}t}\left(X^2+Y^2\right)+2\ddot{\nu}\left(X\dot{Y}-Y\dot{X}\right)=$$

$$-\frac{2fm_0\alpha}{r^{\alpha+2}}\left(X\dot{X}+Y\dot{Y}+Z\dot{Z}\right)+2\left(\frac{\partial R^*_{n-1}}{\partial X}\dot{X}+\frac{\partial R^*_{n-1}}{\partial Y}\dot{Y}+\frac{\partial R^*_{n-1}}{\partial Z}\dot{Z}\right),$$

or

$$\mathrm{d}\left(\dot{X}^2+\dot{Y}^2+\dot{Z}^2\right)-\dot{\nu}^2\mathrm{d}\left(X^2+Y^2\right)+2\ddot{\nu}\left(X\mathrm{d}Y-Y\mathrm{d}X\right)=$$

$$=2fm_0\mathrm{d}\left(\frac{1}{r^\alpha}\right)+\left(\frac{\partial R^*_{n-1}}{\partial X}\mathrm{d}X+\frac{\partial R^*_{n-1}}{\partial Y}\mathrm{d}Y+\frac{\partial R^*_{n-1}}{\partial Z}\mathrm{d}Z\right). \quad (7)$$

The relation (7) can be integrated only in the case in which both its sides are total differentials of functions which do not depend explicitly on the time t. This is possible, for instance, if $\dot{\nu}=\omega=\mathrm{const}$, (hence $\ddot{\nu}=0$) while R^*_{n-1} does not depend on t, that is, $R^*_{n-1}\equiv R^*_{n-1}(X,Y,Z)$. Then we get

$$\mathrm{d}\left(\dot{X}^2+\dot{Y}^2+\dot{Z}^2\right)-\omega^2\mathrm{d}\left(X^2+Y^2\right)=2fm_0\mathrm{d}\left(\frac{1}{r^\alpha}\right)+2\mathrm{d}R^*_{n-1}, \quad (8)$$

which leads to

$$\dot{X}^2+\dot{Y}^2+\dot{Z}^2=\omega^2\left(X^2+Y^2\right)+\frac{2fm_0}{r^\alpha}\mathrm{d}+2(R^*_{n-1}+h), \quad (9)$$

where h is an arbitrary constant.

Grebenicov (1998) gave the conditions under which the integral (9) does exist in the Newtonian restricted many-body problem. Gadomski et al. (1997) provided, for a frame originated in P_0, and for initial conditions symmetrical with respect to P_0, the motion equations for any of the points P_1, ..., P_{n-1}:

$$\begin{aligned}&\ddot{\rho}-\rho\dot{\lambda}^2=-A_{n-1}\rho^{-(\alpha+1)},\\&\rho\ddot{\lambda}+2\dot{\rho}\dot{\lambda}=0,\end{aligned} \quad (10)$$

where ρ, λ stand for the polar coordinates of any of the points P_1, ..., P_{n-1}.

With the initial conditions

$$\rho(0)=a_0,\ \dot{\rho}(0)=0,\ \lambda_k(0)=\frac{2\pi(k-1)}{n-1},\ (\lambda_1\equiv\lambda),\ \dot{\lambda}_k(0)=\omega$$

we can choose ω, such that the function

$$\rho(t)=a_0,\ \ \lambda(t)=\omega t, \quad (11)$$

is a solution of (10). In this case, the bodies P_1, ..., P_{n-1} move on circular orbits (of radius a_0) around P_0, with constant angular velocity ω. Indeed, substituting (11) in equations (10), these ones are satisfied for

$$\omega^2 = \frac{A_n}{a_0^{\alpha+2}}, \quad (12)$$

where $A_{n-1} = f\alpha\left(m_0 + \frac{m}{2^{\alpha+1}}\sum_{k=2}^{n-1}\left(\sin\frac{\pi(k-1)}{n-1}\right)^{-\alpha}\right)$.

In the uniformly rotating (with angular velocity ω = const.) frame P_0XYZ, the bodies P_1, ... ,P_{n-1} are fixed, their coordinates X_k, Y_k, Z_k being

$$X_k = a_0\cos\frac{2\pi(k-1)}{n-1},\ Y_k = a_0\sin\frac{2\pi(k-1)}{n-1},\ Z_k = 0,\ r_k = a_0, \quad (13)$$

therefore the perturbing function

$$R^*_{n-1}(X,Y,Z) = fm\sum_{k=1}^{n-1}\left(\frac{1}{\Delta_k^\alpha} - \frac{XX_k + YY_k + ZZ_k}{r_k^{\alpha+2}}\right) \quad (14)$$

does no longer depend explicitly on t, that is,

$$R^*_{n-1} = R^*_{n-1}(X,Y,Z). \quad (15)$$

Equations (5) acquire the form

$$\ddot{X} - 2\omega\dot{Y} = \omega^2 X - \frac{fm_0\alpha X}{r^{\alpha+2}} + \frac{\partial R^*_{n-1}}{\partial X},$$
$$\ddot{Y} + 2\omega\dot{X} = \omega^2 Y - \frac{fm_0\alpha Y}{r^{\alpha+2}} + \frac{\partial R^*_{n-1}}{\partial Y}, \quad (16)$$
$$\ddot{Z} = \frac{fm_0\alpha Z}{r^{\alpha+2}} + \frac{\partial R^*_{n-1}}{\partial Z}.$$

Multiplying them respectively by $2\dot{X}$, $2\dot{Y}$, $2\dot{Z}$, and adding them together, we get

$$\mathrm{d}\left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2\right) = \omega^2\mathrm{d}\left(X^2 + Y^2\right) + 2fm_0\mathrm{d}\left(\frac{1}{r^\alpha}\right) + 2\mathrm{d}R^*_{n-1}. \quad (17)$$

From (17) we obtain the Jacobi integral for the restricted circular (n+1)-body problem for every attraction law of the type $F_{ks} \sim \Delta_{ks}^{-(\alpha+1)}$ in the form

$$\frac{1}{2}\left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2\right) = \frac{\omega^2}{2}\left(X^2 + Y^2\right) + \frac{fm_0}{r^\alpha} + R^*_{n-1} + h. \quad (18)$$

Considering the conditions for the polygon P_1,...,P_{n-1} to be symmetrically placed in the frame P_0XYZ,

$$\sum_{k=1}^{n-1} X_k = \sum_{k=1}^{n-1} Y_k = 0. \tag{19}$$

we lastly obtain the final formula for the Jacobi integral in the restricted circular (n+1)-body problem with an attraction law depending on an arbitrary power of the reciprocal distances:

$$\frac{1}{2}\left(\dot{X}^2+\dot{Y}^2+\dot{Z}^2\right)=\frac{\omega^2}{2}\left(X^2+Y^2\right)+\frac{fm_0}{r^\alpha}+fm\sum_{k=1}^{n-1}\frac{1}{\Delta_k^\alpha}+h. \tag{20}$$

As in the case of the Newtonian restricted circular many-body problem, (20) shows that the relative total energy of the point P in the frame P_0XYZ is constant and equal to h. It is composed by the relative kinetic energy, the relative centrifugal potential energy, and the relative potential energy.

To end, we shall write the equation of that surface (analogous to Hill's surface in the restricted circular three-body problem; e.g. Duboshin 1964, 1975) for which, in every its point, the moving point P has zero relative velocity. Obviously, it is expressed via the functional equation

$$\frac{\omega^2}{2}\left(X^2+Y^2\right)+\frac{fm_0}{r^\alpha}+fm\sum_{k=1}^{n-1}\frac{1}{\Delta_k^\alpha}+h=0. \tag{21}$$

The study of equation (21) constitutes another, independent, problem. Here we limit ourselves to say that this surface is symmetric with respect to the plane P_0XY.

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Received on 10 February 1998

EQUILIBRIUM POINTS IN THE RESTRICTED FOUR-BODY PROBLEM: SUFFICIENT CONDITIONS FOR LINEAR STABILITY

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Abstract: The motion of a negligible mass in the gravitational field generated by a collinear configuration of three bodies (of masses $m_0 \neq m_1 = m = m_2$) is being studied. Six equilibria are found: four on the straight line defined by the three finite masses, and two which form isosceles triangles with the equal masses. The linear stability of these equilibria is investigated.

Key words: celestial mechanics – restricted four-body problem – stability

1. INTRODUCTION

Many investigations have been devoted to the classical model of the restricted three-body problem (see, e.g., Poincaré 1892; Wintner 1941; Szebehely 1967; Markeev 1978). On the other hand, it was proved (Elmabsout 1988; Grebenicov 1997) that in the general Newtonian n-body problem ($n \geq 3$, all n masses being nonzero) there exists a set of initial conditions leading to circular motions of the n–1 bodies around the n-th one. Grebenicov (1998) suggested a new dynamical model called the restricted circular (n+1)-body problem ($n \geq 3$). The general form of the sufficient conditions of existence of equilibrium points, which are expressed by nonlinear irrational equations, were given by Gadomski and Grebeñikov (1998).

In this paper we study the equilibrium solutions of the restricted circular (n+1)-body problem for $n = 3$ (the case $n = 2$ for the barycentric model was studied by Lagrange).

2. EQUILIBRIUM SOLUTIONS

The differential equations which describe the motion of the point P (of negligible mass) in the gravitational field of the points P_0 (of mass $m_0 \neq 0$) and P_1, P_2 (of masses $m_1 = m_2 = m \neq 0$), in the rotating barycentric Cartesian frame P_0xyz, have the form (Grebenicov 1998):

$$\frac{d^2x}{dt^2} - 2\omega_3\frac{dy}{dt} = \frac{\partial\Omega_3}{\partial x},$$
$$\frac{d^2y}{dt^2} - 2\omega_3\frac{dx}{dt} = \frac{\partial\Omega_3}{\partial y}, \qquad (1)$$
$$\frac{d^2z}{dt^2} = \frac{\partial\Omega_3}{\partial z},$$

where

$$\omega_3^2 = f\left(m_0 + m/4\right), \qquad (2)$$

$$\Omega_3\left(x,y,z\right) = \frac{\omega_3^2}{2}\left(x^2+y^2\right) + f\left[\frac{m_0}{\Delta_0} + \frac{m}{\Delta_1} + \frac{m}{\Delta_2}\right], \qquad (3)$$

$$\Delta_0^2 = x^2+y^2+z^2, \quad \Delta_1^2 = \left(x-1\right)^2+y^2+z^2, \quad \Delta_2^2 = \left(x+1\right)^2+y^2+z^2, \qquad (4)$$

and $f\,(=1)$ stands for the Newtonian gravitational constant.

According to the definition of an equilibrium point (e.g. Abalakin et al. 1976), system (1) must be written in the normal Cauchy form (e.g. Stepanov 1953):

$$\frac{dZ}{dt} = F\left(Z\right), \qquad (5)$$

where the 6-dimensional vectors Z and F have the form:

$$Z^T = (Z_1, Z_2, Z_3, Z_4, Z_5, Z_6),\ F^T = (T_1, T_2, T_3, T_4, T_5, T_6),$$

with

$$\begin{cases} F_1 = Z_4, \\ F_2 = Z_5, \\ F_3 = Z_6, \\ F_4 = 2\omega_3 Z_5 + \omega_3^2 Z_1 - \dfrac{m_0 Z_1}{\Delta_0^3} - m\left[\dfrac{Z_1-1}{\Delta_1^3} + \dfrac{Z_1+1}{\Delta_2^3}\right], \\ F_5 = 2\omega_3 Z_4 + \omega_3^2 Z_2 - \dfrac{m_0 Z_2}{\Delta_0^3} - m\left[\dfrac{Z_2}{\Delta_1^3} + \dfrac{Z_2}{\Delta_2^3}\right], \\ F_6 = -\dfrac{m_0 Z_3}{\Delta_0^3} - m\left[\dfrac{Z_3}{\Delta_1^3} + \dfrac{Z_3}{\Delta_2^3}\right], \\ \Delta_0^2 = Z_1^2 + Z_2^2 + Z_3^2, \\ \Delta_1^2 = \left(Z_1 - 1\right)^2 + Z_2^2 + Z_3^2, \\ \Delta_2^2 = \left(Z_1 + 1\right)^2 + Z_2^2 + Z_3^2. \end{cases} \qquad (6)$$

Every solution of the vector equation

$$F(Z)=0 \tag{7}$$

if such solutions do exist, determines one equilibrium point of the system (5).

The third and sixth equations of the system (7) are satisfied, when $Z_3=Z_6=0$ for arbitrary Z_1, Z_2, Z_4, Z_5 and therefore, instead of equation (7), we have to solve only the equations:

$$F_1=F_2=F_4=F_5=0. \tag{8}$$

It is impossible to find a compact analytic form for the solutions of (8), therefore we used two methods:

1. The method of power series relative to m / m_0 (if $m < m_0 = 1$) and m_0 /m (if $m > m_0$) by using the symbolic computation system Mathematica.

2. Newton's iterative method for given values m / m_0, which was more effective.

We present some examples. On the P_0x-axis there exist four equilibrium points (see Fig. 1, points P - with the coordinates $(x_1^*,0)$, $(x^*_2,0)$, $(-x_1^*, 0)$, $(-x_2^*,0)$).

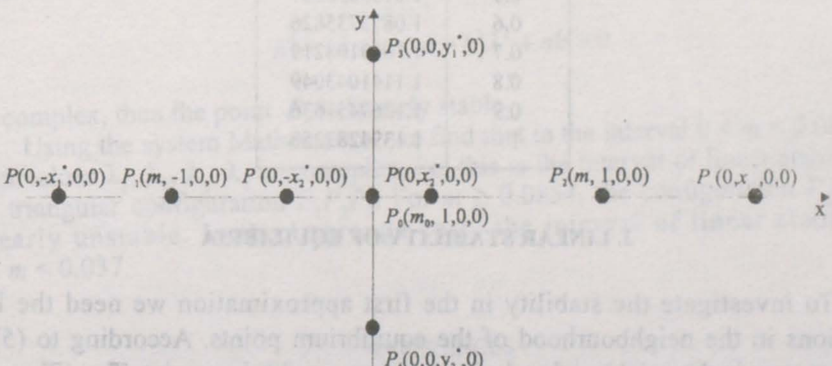

Fig. 1. – Equilibria of the restricted four-body problem.

On the P_0y-axis there exist two equilibrium points P_3, P_4: $(0, y_1^*)$, $(0, y_2^*)$. The quantities y_1^*, y_2^* are the corresponding solutions of the equations

$$\omega_3^2 y^* - \frac{1}{y^{*2}} - \frac{2my^*}{\left(y^{*2}+1\right)^{3/2}} = 0,$$

$$\omega_3^2 y^* + \frac{1}{y^{*2}} - \frac{2my^*}{\left(y^{*2}+1\right)^{3/2}} = 0,$$

It is easy to prove that $y_2^* = -y_1^*$. The triangles $P_1P_2P_3$ and $P_1P_2P_4$ are isosceles, not equilateral, as in the case of the restricted three-body problem. The table below lists the values y_1^* as a function of m / m_0.

Table 1

m / m_0	y_1^*
0.01	1.0015229380
0.02	1.0030443433
0.03	1.0045617150
0.04	1.0060823784
0.05	1.0075989201
0.06	1.0091137536
0.07	1.0106268358
0.08	1.0121381246
0.09	1.0136475778
0.1	1.0151551543
0.2	1.0301189216
0.3	1.0448530923
0.4	1.0593245942
0.5	1.0735056257
0.6	1.0873735826
0.7	1.1009108219
0.8	1.1141043049
0.9	1.1269451636
1	1.1394282250

3. LINEAR STABILITY OF EQUILIBRIA

To investigate the stability in the first approximation we need the linear equations in the neighbourhood of the equilibrium points. According to (5), the linear system in the neighbourhood of an arbitrary equilibrium point $Z^* = (Z^*_1, \ldots, Z^*_6)$ has the form:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au, \tag{9}$$

where

$$u = Z - Z^*$$
$$A = (a_{ik})_{i,k=1,\ldots,6} \tag{10}$$

It can be proved, that the "straight line"(collinear) configurations are linearly unstable for any value m, as in the Lagrange case of the three-body problem. So, let us study the triangular configuration P_1,P_2,P_3. In this case the elements of the matrix A are

$$a_{14} = a_{25} = a_{36} = 1,\ a_{41} = a,\ \ a_{45} = 2\omega_3 = -a_{54},\ \ a_{52} = b,\ a_{63} = c,$$

the other a_{ik} being zero. The quantities a, b, c read

$$a = \omega_3^2 + \frac{6m}{\left(1 + y_1^{*2}\right)^{5/2}} - \frac{2m}{\left(1 + y_1^{*2}\right)^{3/2}} - \frac{1}{y_1^{*3}}$$

$$b = \omega_3^2 + \frac{6my_1^{*2}}{\left(1 + y_1^{*2}\right)^{5/2}} - \frac{2m}{\left(1 + y_1^{*2}\right)^{3/2}} + \frac{2}{y_1^{*3}}$$

$$c = -\frac{2m}{\left(1 + y_1^{*2}\right)^{3/2}} - \frac{1}{y_1^{*3}} .$$

Then the eigenvalues of the matrix A are defined by the algebraic equation

$$\left(\lambda^2 - c\right)\left(\lambda^4 + \left(4 + m - a - b\right)\lambda^2 + ab\right) = 0. \tag{11}$$

Because $c<0$ for arbitrary values m, two eigenvalues λ_5 and λ_6 are always imaginary. If all roots of the equation

$$\lambda^4 + \left(4 + m - a - b\right)\lambda^2 + ab = 0 \tag{12}$$

are complex, then the point P_3 is linearly stable.

Using the system Mathematica, we find that in the interval $0 < m < 0.0853$ the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are complex, i.e. this is the interval of linear stability for the triangular configuration $P_1P_2P_3$. For $m \geq 0.0854$, the configuration $P_1P_2P_3$ is linearly unstable. In the Lagrange case, the interval of linear stability is $0 < m < 0.037$.

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Received on 10 February 1998

PRELIMINARY LOCATION OF THE EQUILIBRIA OF THE TWO-BODY PROBLEM IN EINSTEIN'S PN FIELD

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Abstract. The equilibria of the two-body problem associated to a spherical PN field with Einsteinian parametrization are being tackled. The location of the equilibria in a phase plane determined by the radius vector and angular momentum is sketched. For a given angular momentum, one finds at most two distinct relative equilibria. One also finds a distance range deprived of equilibria.

Key words: celestial mechanics – PN fields – McGehee transformations

1. INTRODUCTION

The qualitative study of the motion of a test particle in a spherical PN field with Einsteinian parametrization ($\alpha = 1$, $\beta = 1$, $\gamma = 1$; see e.g. Soffel 1989) was tackled by Şelaru et al. (1997). In suitably chosen units, and neglecting the terms of order $c^{-n}(n \geq 4)$, where c is the speed of light, the Hamiltonian which features the problem reads

$$H(q,p) = \frac{1}{2}|p|^2 - |q|^{-1} - \frac{1}{8c^2}|p|^4 - \frac{3}{2c^2}|p|^2|q|^{-1} + \frac{1}{2c^2}|q|^{-2}, \tag{1}$$

in which $q = (q_1, q_2) \in \Re \backslash \{(0,0)\}$ is the position vector of the particle, whereas $p = (p_1, p_2) \in \Re^2$ is the conjugate momentum vector.

The motion being planar (due to the isotropy of space; e.g. Soffel 1989), we have chosen the motion plane to be the (q_1, q_2)-plane. This choice led to the expression (1) of the Hamiltonian.

Şelaru et al. (1997) used McGehee-type transformations of the second kind (McGehee 1974) to regularize the equation of motion and to blow up the collision singularity. They described the flow on the collision manifold M_0. Next they defined the infinity manifold M_∞ and depicted the flow on it. (We have to mention that these special flows have no physical significance, but – due to the continuity of solutions with respect to the initial conditions – they provide valuable information about the orbits which neighbor collision or infinity.)

In this note we tackle the problem of the relative equilibria situated outside M_0 and M_∞. We find that, for a given angular momentum, there can exist at most two equilibrium orbits. We also find a distance range deprived of relative equilibria.

2. EQUATIONS OF MOTION

Starting from the canonical equations of motion, Şelaru et al. (1997) used successively the following McGehee-type transformations of the second kind:

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = r \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \tag{2}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = r^{-1/2} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{3}$$

$$\mathrm{d}t = r^{5/2}\mathrm{d}\tau, \tag{4}$$

which all are real analytic diffeomorphisms. Under these transformations, the motion equations become

$$\begin{aligned} r' &= rxA, \\ \theta' &= yA, \\ x' &= (x^2/2 + y^2)A - B, \\ y' &= -xyA/2, \end{aligned} \tag{5}$$

where we kept by abuse the same notation for the new functions of the timelike variable τ. In (5) we abridged

$$\begin{aligned} A(r,x,y) &= r - \left(x^2 + y^2 + 6\right)/\left(2c^2\right), \\ B(r,x,y) &= r + \left[3\left(x^2 + y^2\right) - 2\right]/\left(2c^2\right). \end{aligned} \tag{6}$$

The problem admits the first integrals of energy and angular momentum, which respectively read

$$H(q,p) = h, \tag{7}$$

$$q_1p_2 - q_2p_1 = K, \tag{8}$$

where h and k are the respective constants. Under the transformations (2)-(3), these integrals acquire the form

$$r\left(x^2+y^2-2\right)/2-\left[\left(x^2+y^2+6\right)^2-40\right]/\left(8c^2\right)=hr^2, \tag{9}$$

$$\sqrt{r}y=K. \tag{10}$$

Observe that the equations of motion are regular now, and the collision singularity is replaced by a manifold M_0 pasted on the phase space. Since M_0 is invariant under the flow, the phase space extends smoothly to this boundary (it is the same for the energy relation).

The (fictitious) flows on the collision and infinity manifolds were described by Şelaru et al. (1997). To tackle the study of the motion outside these manifolds, a first step is represented by the searching for equilibria. Of course, we refer to critical points with $r \neq 0$ (according to (4), the collision manifold is formed only by equilibria in the full phase space of McGehee-type coordinates).

3. LOCATION OF EQUILIBRIA

Let us start from the vector field (5), and suppose that $A = 0$ leads to equilibrium. (By the way, this would mean an absolute equilibrium, namely rest, if we take into account the second equation (5)). Then $r=r_e=\left(x_e^2+y_e^2+6\right)/\left(2c^2\right)$, and by the third equation (5), we obtain

$$x'=-2\left(x_e^2+y_e^2+1\right)/c^2<0, \tag{11}$$

a contradiction. Therefore $A=0$ does not define an equilibrium state. This necessarily implies that equilibria occur for $x = 0$.

Now let us go a step further. Observe that θ does not appear explicitly in the vector field (5). Consequently we may factorize the flow to S^1, reducing in this way the dimension of the phase space from 4 to 3. A critical point in this reduced phase space (RPS) will represent a (topological) circle in full phase space if $y_e \neq 0$. The physical orbits corresponding to these relative equilibria are circular. If $y_e = 0$, the equilibrium point in RPS remains a point in full phase space. In terms of physical motion, this means rest of the particle with respect to the field-generating body.

Consider hence the possibility of the existence of relative equilibria for $x = 0$. The third equation (5) leads to

$$y_e^4-\left(2c^2r_e-9\right)y_e^2+\left(2c^2r-2\right)=0. \tag{12}$$

Examining this equation, we observe that in the (r, y)-plane we have

(i) $r_e \in (0, 1/c^2)$: 2 equilibria;

(ii) $r_e = 1/c^2$: 1 equilibrium (rest; $y_e = 0$);

(iii) $r_e \in \left(1/c^2, \left(11+4\sqrt{2}\right)/\left(2c^2\right)\right)$: no equilibria;

(iv) $r_e \in \left(11+4\sqrt{2}\right)/\left(2c^2\right)$: 2 equilibria $\left(y_e = \pm\sqrt{2\sqrt{2}+1}\right)$;

(v) $r_e \in \left(\left(11+4\sqrt{2}\right)/\left(2c^2\right), \infty\right)$: 4 equilibria.

On this basis, one can sketch the location of the equilibria in the (r, y)-plane (Fig. 1).

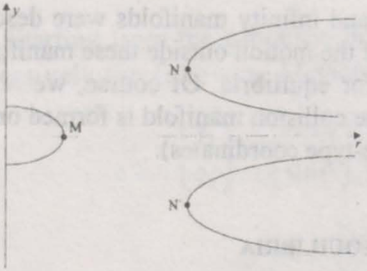

Fig. 1 – Location of the equilibria in the (r, y)-plane. The point M represents the case (ii); the points N, N′ represent the case (iv).

The informations which result from the combination of the algebraic equation (12) with the angular momentum integral (10) corroborate those provided by Fig. 1. This combination yields

$$2c^2 r_e^3 - 2\left(1+c^2K^2\right)r_e^2 + 9K^2 r_e + K^4 = 0. \tag{13}$$

It is easy to deduce that, in physical terms, equation (13) provides at most two real positive roots (confounded or not).

Our preliminary endeavor stops here. To go deeper into the investigation focused on the equilibria, several ways are to be followed:

– analysis of equation (13) for the whole range of values of the angular momentum;

– linearization of the motion equations (5);

– study of the nonlinear stability of the equilibria by means of the technique proposed by Zombro and Holmes (1993). Performing such studies means new steps towards the understanding of such an interesting mathematical and physical problem.

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Received on 10 February 1998

RECURRENT POWER SERIES SOLUTION OF THE n-BODY PROBLEM ASSOCIATED TO A QUASIHOMOGENEOUS POTENTIAL

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Abstract. Using Steffensen's method, a recurrent power series solution is given for the n-body problem associated to a quasihomogeneous potential of form $W = U + V$, where U and V are homogeneous functions of degree $-a$ and $-b$ respectively, with $1 \leq a \leq b$. The application to numerical integration is also pointed out.

Key words: celestial mechanics, generalized fields, power series, numerical integration.

1. INTRODUCTION

In 1956–1957 some articles have been published by Steffensen (1956a, 1956b, 1957), describing the solution of both the restricted and the general three-body problem in term of power series in time. The method proposed by Steffensen is particularly well adapted to computers. The method is made practical by the introduction of a certain number of auxiliary dependent variables, wich transform the system of differential equations where all denominators have been removed, as well as the powers r^3. Steffensen calls his system of 'second degree', because in the final form, only products of two dependent variables appear. This form is particularly well-prepared for the substitution of power series and the identification of equal powers in t. In several of his papers Steffensen has also given convergence criteria for the series. The application of the series is particularly interesting because the square roots are completely avoided in the computations and the number of divisions is reduced to a minimum. The reason why the method is so well adapted to automatic computers is that the calculation of all the coefficients of the power series is done in a recurrent way; for each order, the coefficients are functions of all the precedingly computed coefficients.

This method has been effectively used by several authors, such as Rabe (1961), Deprit (1965), Broucke (1971), Pál and Szenkovits (1996) for the numerical integration of the restricted three-body problem and the general n-body problem on computers, and it appears that the results are superior, both in speed and in precision, to those obtained with most of the classical numerical integration methods.

The goal of this paper is to give the recurrent power series solution for the n-body problem associated to a quasihomogeneous potential of form $W = U + V$, where U and V are homogeneous functions of degree $-a$ and $-b$ respectively, with $1 \leq a \leq b$.

2. THE n-BODY PROBLEM ASSOCIATED TO A QUASIHOMOGENEOUS POTENTIAL

The n-body problem associated to the quasihomogeneous potential can be formulated as follows (see Diacu, 1996). Consider n particles of masses $m_i > 0$ in the Euclidean space E^3, having coordinates $\mathbf{q}_i = (q_i^1, q_i^2, q_i^3)$, $i = 1, 2, \ldots, n$, in an inertial reference system. Let $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n) \in \mathbf{R}^{3n}$ be the configuration of the system of particles and define the *quasihomogeneous* potential $W = U + V$, where

$$U:\ \mathbf{R}^{3n}\setminus \Delta \to \mathbf{R}_+,\ U(\mathbf{q}) = \sum_{1\le i<j\le n} \alpha(m_i, m_j)\, q_{ij}^{-a},$$

$$V:\ \mathbf{R}^{3n}\setminus \Delta \to \mathbf{R}_+,\ V(\mathbf{q}) = \sum_{1\le i<j\le n} \beta(m_i, m_j)\, q_{ij}^{-b}$$

are homogeneous functions of degree $-a$ and $-b$ respectively, with $1 \leq a \leq b$. In these potentials $q_{ij} = |\mathbf{q}_i - \mathbf{q}_j|$ is the Euclidean distance between particles i and j, Δ denotes the collision-ejection set

$$\Delta = \bigcup_{1\le i<j\le n} \left\{ \mathbf{q} \middle| \mathbf{q}_i = \mathbf{q}_j \right\}$$

and α, β are symmetric positive functions of the masses, i.e. such that $\alpha(m_i, m_j) = \alpha(m_j, m_i) = \alpha_{ij} > 0$, and $\beta(m_i, m_j) = \beta(m_j, m_i) = \beta_{ij} > 0$, for all $1 \leq i < j \leq n$.

The equations of motion are given by the system

$$\begin{cases} \dot{\mathbf{q}} = M^{-1}\mathbf{p} \\ \dot{\mathbf{p}} = \nabla W(\mathbf{q}), \end{cases} \tag{1}$$

where

$M = \mathrm{diag}(m_1, m_1, m_1, m_2, m_2, m_2, \ldots, m_n, m_n, m_n)$,

$\nabla = \left(\partial/\partial \mathbf{q}_1, \partial/\partial \mathbf{q}_2, \ldots, \partial/\partial \mathbf{q}_n \right)$

is the gradient operator and

$$\mathbf{p} = M\dot{\mathbf{q}},\ \mathbf{p} = \left(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\right) \in \mathbf{R}^{3n}$$

denotes the *momentum* of the system. In case $a = b = 1$ and $\alpha(m_i, m_j) = \beta(m_i, m_j) = (G/2)\, m_i m_j$, where G is the gravitational constant, we are in the classical Newtonian n-body problem.

3. RECURRENT POWER SERIES SOLUTION

The equations of motion (1) are equivalents with the:

$$\begin{cases} m_i \dot{\mathbf{q}}_i = \mathbf{p}_i \\ \dot{\mathbf{p}}_i = \sum\limits_{\substack{j=1 \\ j\neq i}}^{n} \left(a\alpha_{ij} q_{ij}^{-a-2} + b\beta_{ij} q_{ij}^{-b-2}\right)\left(\mathbf{q}_j - \mathbf{q}_i\right), \end{cases} \quad i = 1, 2, ..., n \qquad (2)$$

The central idea of Steffensen's method is to introduce auxiliary variables, which help us to eliminate quantities $q_{ij}^{a+2}, q_{ij}^{b+2}$ from the denominators of the right hand side expressions in equations (2). Let it be:

$$\begin{cases} a_{ij} = q_{ij}^{-a-2} \\ b_{ij} = q_{ij}^{-b-2} \end{cases}, \quad 1 \le i < j \le n \qquad (3)$$

Using these new variables introduced in (3), the equations of motion (2) have the new form

$$\begin{cases} m_i \dot{\mathbf{q}}_i = \mathbf{p}_i \\ \dot{\mathbf{p}}_i = \sum\limits_{\substack{j=1 \\ j\neq i}}^{n} \left(a\alpha_{ij} a_{ij} + b\beta_{ij} b_{ij}\right)\left(\mathbf{q}_j - \mathbf{q}_i\right), \end{cases} \quad i = 1, 2, ..., n \qquad (4)$$

Equations (4) can be completed with the next relations between old and new variables:

$$\begin{cases} q_{ij}\dot{q}_{ij} = \left(\mathbf{q}_j - \mathbf{q}_i\right)\left(\dot{\mathbf{q}}_j - \dot{\mathbf{q}}_i\right) \\ q_{ij}\dot{a}_{ij} = -(a+2)a_{ij}\dot{q}_{ij} \\ q_{ij}\dot{b}_{ij} = -(b+2)b_{ij}\dot{q}_{ij} \end{cases}, \quad 1 \le i < j \le n \qquad (5)$$

Equations (4) and (5) constitute a differential system which determine the variables $\mathbf{q}_i$, $\mathbf{p}_i$, q_{ij}, a_{ij}, b_{ij}, $1 \le i < j \le n$. This system cannot be integrated with exact methods. We can obtain the solutions of this system using power series:

$$\mathbf{q}_i = \sum_{k=1}^{\infty} \mathbf{Q}_{ik} t^{k-1}, \quad \mathbf{p}_i = \sum_{k=1}^{\infty} \mathbf{P}_{ik} t^{k-1},$$

$$q_{ij} = \sum_{k=1}^{\infty} Q_{ijk} t^{k-1}, \quad a_{ij} = \sum_{k=1}^{\infty} A_{ijk} t^{k-1}, \quad b_{ij} = \sum_{k=1}^{\infty} B_{ijk} t^{k-1}, \tag{6}$$

where the coefficients $\mathbf{Q}_{ik}$, $\mathbf{P}_{ik} \in \mathbf{R}^3$, Q_{ijk}, A_{ijk}, $B_{ijk} \in \mathbf{R}$, $1 \le i < j \le n$, $k \ge 1$ have to be determined.

Substituting the power series (6) in equations (4, 5) after identification of equal powers in t, one obtains the next recurrence relations, to determine the unknown coefficients:

$$\begin{aligned}
& k m_i \mathbf{Q}_{i,k+1} = \mathbf{P}_{ik} \\
& k\mathbf{P}_{i,k+1} = \sum_{\substack{j=1 \\ j \ne i}}^{n} \left[a\alpha_{ij} \sum_{\substack{p+q=k+1 \\ p=1,\dots,k}} A_{ijp}\left(\mathbf{Q}_{jq} - \mathbf{Q}_{iq}\right) + b\beta_{ij} \sum_{\substack{p+q=k+1 \\ p=1,\dots,k}} B_{ijp}\left(\mathbf{Q}_{jq} - \mathbf{Q}_{iq}\right) \right] \\
& kQ_{ij1} Q_{ij,k+1} = - \sum_{\substack{p+q=k+1 \\ p=2,\dots,k}} q Q_{ijp} Q_{ij,q+1} + \sum_{\substack{p+q=k+1 \\ p=1,\dots,k}} q\left(\mathbf{Q}_{jp} - \mathbf{Q}_{ip}\right)\left(\mathbf{Q}_{j,q+1} - \mathbf{Q}_{i,q+1}\right) \\
& kQ_{ij1} A_{ij,k+1} = - \sum_{\substack{p+q=k+1 \\ p=2,\dots,k}} q Q_{ijp} A_{ij,q+1} - (a+2) \sum_{\substack{p+q=k+1 \\ p=1,\dots,k}} q A_{ijp} Q_{ij,q+1} \\
& kQ_{ij1} B_{ij,k+1} = - \sum_{\substack{p+q=k+1 \\ p=2,\dots,k}} q Q_{ijp} B_{ij,q+1} - (b+2) \sum_{\substack{p+q=k+1 \\ p=1,\dots,k}} q B_{ijp} Q_{ij,q+1}
\end{aligned} \tag{7}$$

Start coefficients $\mathbf{Q}_{i1}$, $\mathbf{P}_{i1} \in \mathbf{R}^3$, Q_{ij1}, A_{ij1}, $B_{ij1} \in \mathbf{R}$, $1 \le i < j \le n$ are obtained from the initial conditions given for $t = 0$, $\mathbf{q}_{i0} = \mathbf{q}_i(0)$, $\mathbf{p}_{i0} = \mathbf{p}_i(0) \in \mathbf{R}^3$, $i = 1, 2, \dots, n$. These coefficients are:

$$\begin{aligned}
& \mathbf{Q}_{i1} = \mathbf{q}_{i0}, \mathbf{P}_{i1} = \mathbf{p}_{i0}, i = 1, 2, \dots, n \\
& Q_{ij1} = |\mathbf{Q}_{i1} - \mathbf{Q}_{j1}|, \; A_{ij1} = Q_{ij1}^{-a-2}, \; \mathrm{B}_{ij1} = Q_{ij1}^{-b-2}, \; 1 \le i < j \le n.
\end{aligned} \tag{8}$$

Start coefficients (8) calculated, one calculates all coefficients of Taylor series (6), with the index $k + 1$, step by step using the recurrent relations (7).

5. APLICATIONS TO NUMERICAL INTEGRATION

This method is well adapted to automatic computers, since calculation of all the coefficients of the power series is done in a recurrent way; for each order, the coefficients are functions of all the precedingly computed coefficients. In approximate numerical solutions the coefficients are calculated up to the order N, suitable chosen. To verify the computational accuracy, one can uses the *energy integral*

$$T(\mathbf{p}(t)) - W(\mathbf{q}(t)) = h$$

where

$$T : \mathbf{R}^{3n} \to [0, +\infty),\, T(\mathbf{p}) = \frac{1}{2}\sum_{i=1}^{n} m_i^{-1}|\mathbf{p}_i|^2$$

is the *kinetic energy* and *h* is the *energy constant*; and the *momentum integral* .

$$\sum_{i=1}^{n}[\mathbf{q}_i, \mathbf{p}_i] = \mathbf{c} \quad \text{(constant)}.$$

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Received on 12 January 1998

CCD AND PHOTOGRAPHIC OBSERVATIONS OF THE COMET C/1996 B2 (HYAKUTAKE)

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Abstract: About 200 CCD images of the comet C/1996 B2 - HYAKUTAKE were obtained and reduced, using four approaches by PPM stars and one by a GSC star, in March 20, 24, 25 and April 1-st, 1996, in Bucharest. Also, three photographically plates were reduced using PPM stars. The O-C analysis of the astrometric data allowed testing the accuracy of both the observational technique and the reduction method. The variation of the orientation of the comet's tail was also computed.

Key words: comet, appulse, astrometry, CCD

1. INTRODUCTION

Comets, together with asteroids, play an essential role in the knowledge of the origin and evolution of the Solar System. The spectacular and rare visible naked-eye apparitions of a comet in the inner Solar System become a major event for the astronomical community.

C/1996 B2 (Hyakutake) was announced at the end of January 1996, and became one of the "comets of the century" at the end of this century. The close approach opportunities from the Earth allowed favorable conditions of ground-based observations, even for the modest instruments.

More than 400 comets were observed photographically at Bucharest Observatory (Vass, 1994). The double refractor Prin-Merz of the Astronomical Institute in Bucharest has an $F = 6m/D = 38cm$ and works as an astrograph, both photographically and CCD. The plates have a field by 2°×2° and ensure a limiting magnitude 12 (at maximum 30 min exposure time). The CCD has 768×512 pixels, "sees" a 4'×2.5' field and a limiting magnitude 15 (at 15 seconds time of integration). It is used in binning mode 2, 1pixel = 0.62".

2. ASTROMETRICAL OBSERVATIONS

In two cloudless nights we obtained three photographically positions of Hyakutake (March 20-th and 24-th), which were measured using an

ASCORECORD machine, and were reduced with a classical least-squares method, using five-PPM stars. The results are given in Table 1.

Table 1

Photographic positions of the comet C/1996 B2

Date	Topocentric positions		Geocentric positions	
	α_{2000}	δ_{2000}	α_{2000}	δ_{2000}
$1996^y 3^m 20^d.930716$	$14^h\ 52^m\ 21^s.956$	$4°\ 40'\ 13''.04$	$14^h\ 52^m\ 20^s.330$	$4°\ 40'\ 47''.93$
1996 3 20 .950107	14 52 20 .179	4 47 27 .77	14 52 18 .776	4 48 02 .66
1996 3 24 .951746	14 35 33 .646	53 04 50 .37	14 35 30 .493	53 06 30 .85

Since the equipping of the mentioned instrument with the CCD, although various sources have been observed (the Saturnian satellites, asteroid appulses, globular clusters), C/1996 B2 is the first cometary object observed by CCD in Bucharest. His spectacular passing near the Earth in March 1996 offered the opportunity to test this new astrometrical technique on the comet observations.

March 24/25 was the most fruitful night of observation, because of the minimum distance of the comet from the Earth, which produced the greatest proper motion at that time 1'/min (Marsden, 1996), and due to the good meteorological conditions.

The possibility to make astrometrical observations using a CCD was discussed previously (Văduvescu & Vass, 1995). Comparing with the photographical observations, the main problem of the CCD consists in the small field of the receptor ($10.4016'^2$), and consequently in the small density of the catalogue stars in the field (0.025 PPM stars/CCD field). Nevertheless, there are two important advantages of the CCD: the very short integration time (and accordingly a lot of exposures), and the procedure of the measurement of the source's position (such as the Gauss distribution of the light intensities). This method gives good results mainly to diffuse sources (as the nucleus of the comets), comparing to the visual method at the measuring machine for astrometric positions.

In order to plot the path of the comet through the stars, graphical software, *Celestial Maps v.4.5* was used (Văduvescu & Bîrlan, 1996). Both the program and the ephemeris certified the accuracy of the predictions. Thus, in March 24/25 the comet approached four-PPM stars (PPM34602 at 22^h0^m, PPM34601 at 22^h34^m, PPM34595 at 0^h36^m, and PPM34582 at 1^h39^m, all in UT). At these moments, 15, 22, 20 and respectively 26 CCD exposures were made using 5-s times of integration (the stars has V_{ph} = 11.1, 10.1, 10.3 and 8.0 respectively).

The method of reduction uses the *one-star* reference systems (Văduvescu & Vass, 1995), and the orientation of the CCD was solved using *two - PPM stars* in the vicinity of the comet. All the 83 positions of the nucleus were reduced and reported to the Central Bureau for Astronomical Telegrams.

The reduced positions of the comet are presented in Figure 1. We can observe the good agreement between the results obtained by the two methods.

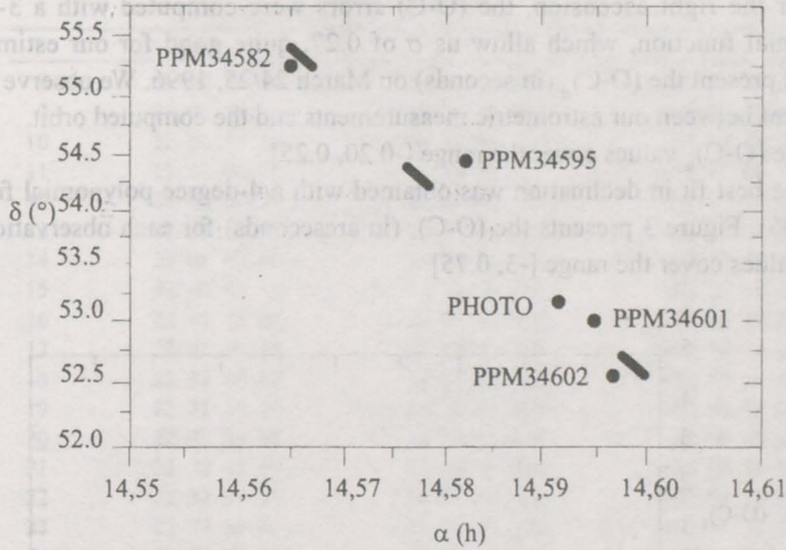

Fig. 1. – Reduced photographic and CCD positions of the comet C/1996 B2

We estimated the possibility to make astrometry using a single reference catalogue star. For this purpose, the (O-C) estimation is a good indicator. We used the comparison of our observations with the computed positions of Smithsonian (B. Marsden, private correspondence). Then, we used a polynomial fit.

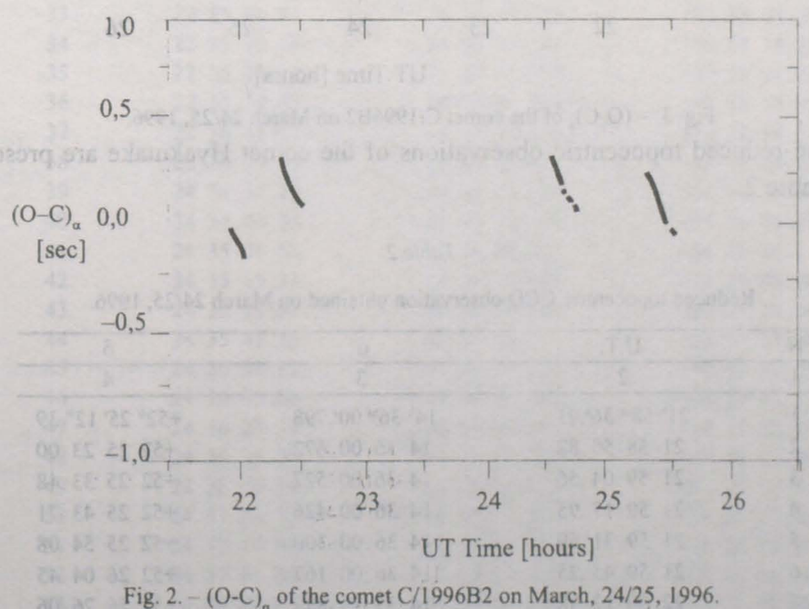

Fig. 2. – $(O\text{-}C)_\alpha$ of the comet C/1996B2 on March, 24/25, 1996.

For the right ascension, the (O-C) errors were computed with a 3-degree polynomial function, which allow us σ of 0.27, quite good for our estimations. Figure 2 present the $(O\text{-}C)_\alpha$ (in seconds) on March 24/25, 1996. We observe a good agreement between our astrometric measurements and the computed orbit.

The $(O\text{-}C)_\alpha$ values cover the range [-0.20, 0.25].

The best fit in declination was obtained with a 4-degree polynomial function (σ = 0.05). Figure 3 presents the $(O\text{-}C)_\delta$ (in arcseconds) for each observation. The (O-C) values cover the range [-3, 0.75].

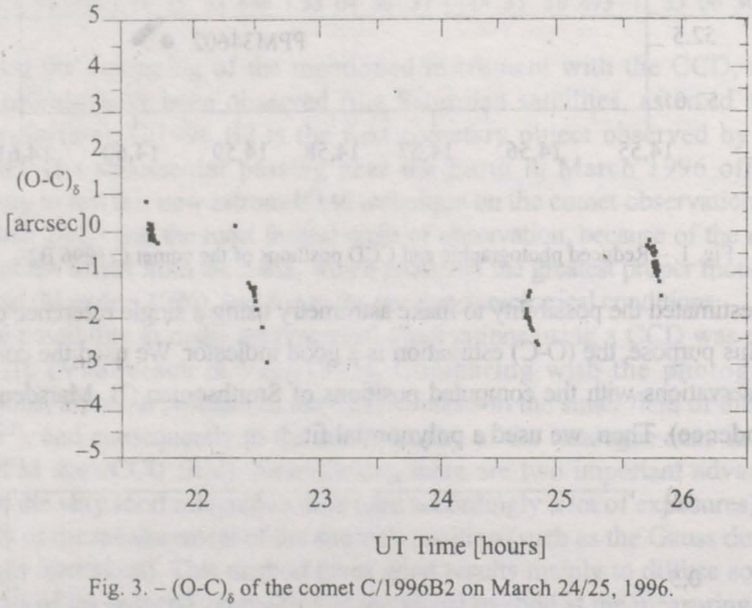

Fig. 3. – $(O\text{-}C)_\delta$ of the comet C/1996B2 on March 24/25, 1996.

The reduced topocentric observations of the comet Hyakutake are presented in the Table 2.

Table 2

Reduced topocentric CCD observation obtained on March 24/25, 1996.

N.	U.T.	α	δ
1	2	3	4
1	$21^h 58^m 36^s.91$	$14^h 36^m 00^s.798$	+52° 25' 12".39
2	21 58 50 .82	14 36 00 .672	+52 25 23 .00
3	21 59 04 .56	14 36 00 .527	+52 25 33 .48
4	21 59 17 .95	14 36 00 .426	+52 25 43 .71
5	21 59 31 .69	14 36 00 .300	+52 25 54 .08
6	21 59 45 .25	14 36 00 .167	+52 26 04 .45
7	22 00 12 .38	14 35 59 .899	+52 26 26 .06

Table 2 (continued)

1	2	3	4
8	22 00 25 .95	14 35 59 .805	+52 26 35 .62
9	22 00 39 .69	14 35 59 .683	+52 26 45 .80
10	22 00 53 .16	14 35 59 .556	+52 26 56 .21
11	22 01 06 .82	14 35 59 .432	+52 27 06 .75
12	22 01 20 .38	14 35 59 .312	+52 27 16 .95
13	22 01 33 .95	14 35 59 .188	+52 27 27 .43
14	22 01 47 .60	14 35 59 .059	+52 27 37 .84
15	22 02 01 .16	14 35 58 .926	+52 27 48 .14
16	22 31 15 .08	14 35 43 .992	+52 50 07 .53
17	22 31 29 .16	14 35 43 .870	+52 50 18 .15
18	22 31 42 .82	14 35 43 .738	+52 50 28 .62
19	22 31 56 .38	14 35 43 .610	+52 50 38 .95
20	22 32 09 .95	14 35 43 .478	+52 50 49 .40
21	22 32 23 .60	14 35 43 .359	+52 50 59 .71
22	22 32 37 .25	14 35 43 .224	+52 51 10 .07
23	22 32 50 .81	14 35 43 .106	+52 51 20 .54
24	22 32 50 .81	14 35 43 .104	+52 51 20 .54
25	22 33 04 .38	14 35 42 .980	+52 51 30 .85
26	22 33 18 .12	14 35 42 .849	+52 51 41 .32
27	22 33 31 .68	14 35 42 .719	+52 51 51 .69
28	22 33 45 .24	14 35 42 .599	+52 52 02 .10
29	22 33 58 .81	14 35 42 .480	+52 52 13 .99
30	22 34 12 .46	14 35 42 .342	+52 52 21 .45
31	22 34 39 .59	14 35 42 .091	+52 52 43 .53
32	22 34 53 .33	14 35 41 .963	+52 52 54 .01
33	22 35 06 .81	14 35 41 .830	+52 53 04 .36
34	22 35 20 .46	14 35 41 .696	+52 53 14 .18
35	22 35 34 .11	14 35 41 .586	+52 53 24 .97
36	22 36 14 .03	14 35 41 .208	+52 53 55 .45
37	22 36 41 .07	14 35 40 .952	+52 54 16 .16
38	22 36 54 .63	14 35 40 .828	+52 54 26 .44
39	24 34 34 .20	14 34 35 .624	+54 24 29 .19
40	24 34 48 .03	14 34 35 .478	+54 24 39 .69
41	24 35 01 .68	14 34 35 .342	+54 24 50 .11
42	24 35 15 .33	14 34 35 .195	+54 25 00 .50
43	24 35 28 .81	14 34 35 .059	+54 25 10 .94
44	24 35 42 .55	14 34 34 .911	+54 25 21 .26
45	24 35 56 .11	14 34 34 .779	+54 25 31 .75
46	24 36 09 .68	14 34 34 .643	+54 25 42 .12
47	24 36 23 .33	14 34 34 .513	+54 25 52 .49
48	24 36 36 .89	14 34 34 .360	+54 26 02 .96
49	24 36 50 .54	14 34 34 .226	+54 26 13 .34
50	24 37 04 .11	14 34 34 .097	+54 26 23 .73
51	24 37 17 .76	14 34 33 .940	+54 26 34 .27
52	24 37 31 .32	14 34 33 .808	+54 26 44 .49
53	24 39 06 .45	14 34 32 .828	+54 27 58 .36

Table 2 (continued)

1	2	3	4
54	24 39 20 .10	14 34 32 .710	+54 28 08 .91
55	24 39 33 .75	14 34 32 .552	+54 28 18 .10
56	24 39 47 .23	14 34 32 .423	+54 28 28 .47
57	24 40 00 .97	14 34 32 .279	+54 28 38 .89
58	24 40 14 .45	14 34 32 .147	+54 28 49 .23
59	25 36 22 .23	14 33 59 .208	+55 11 53 .19
60	25 36 35 .97	14 33 59 .070	+55 12 03 .50
61	25 36 49 .45	14 33 58 .928	+55 12 13 .89
62	25 37 03 .10	14 33 58 .779	+55 12 24 .31
63	25 37 16 .67	14 33 58 .636	+55 12 35 .06
64	25 37 30 .32	14 33 58 .493	+55 12 45 .22
65	25 37 43 .88	14 33 58 .348	+55 12 55 .40
66	25 37 57 .53	14 33 58 .212	+55 13 06 .19
67	25 38 11 .18	14 33 58 .071	+55 13 16 .39
68	25 38 24 .66	14 33 57 .921	+55 13 27 .04
69	25 38 38 .40	14 33 57 .772	+55 13 37 .48
70	25 38 51 .88	14 33 57 .636	+55 13 47 .80
71	25 39 05 .53	14 33 57 .487	+55 13 58 .21
72	25 39 19 .18	14 33 57 .343	+55 14 08 .52
73	25 39 32 .75	14 33 57 .197	+55 14 18 .88
74	25 39 46 .40	14 33 57 .055	+55 14 29 .34
75	25 39 59 .96	14 33 56 .902	+55 14 39 .78
76	25 40 13 .53	14 33 56 .769	+55 14 50 .19
77	25 40 40 .66	14 33 56 .477	+55 15 11 .06
78	25 40 54 .31	14 33 56 .342	+55 15 20 .17
79	25 41 07 .87	14 33 56 .198	+55 15 31 .83
80	25 41 21 .52	14 33 56 .046	+55 15 42 .17
81	25 41 35 .09	14 33 55 .911	+55 15 52 .67
82	25 41 48 .74	14 33 55 .773	+55 16 03 .04
83	25 42 02 .30	14 33 55 .607	+55 16 13 .58

Those observations are very useful in certain cases, such as the objects weakly observed. Thus, we tested for the first time such a method and we make the evaluation of the measurements with our CCD camera.

3. ORIENTATION OF THE TAIL

Figures 4 and 5 present two images of the comet, taken in March 24-th and April 1-st, and viewing in isophotes.

We made measurements of the position angle of the comet's tail using five images for each approach to the PPM stars on March 24/25. First, the position of the nucleus was measured using the centering method described above. After that, using a double representation of the images (in false colors and isophotes), we measured ten position in the tail direction on each image. These absolute coordinates on the images were reduced using linear regression. We obtained in the

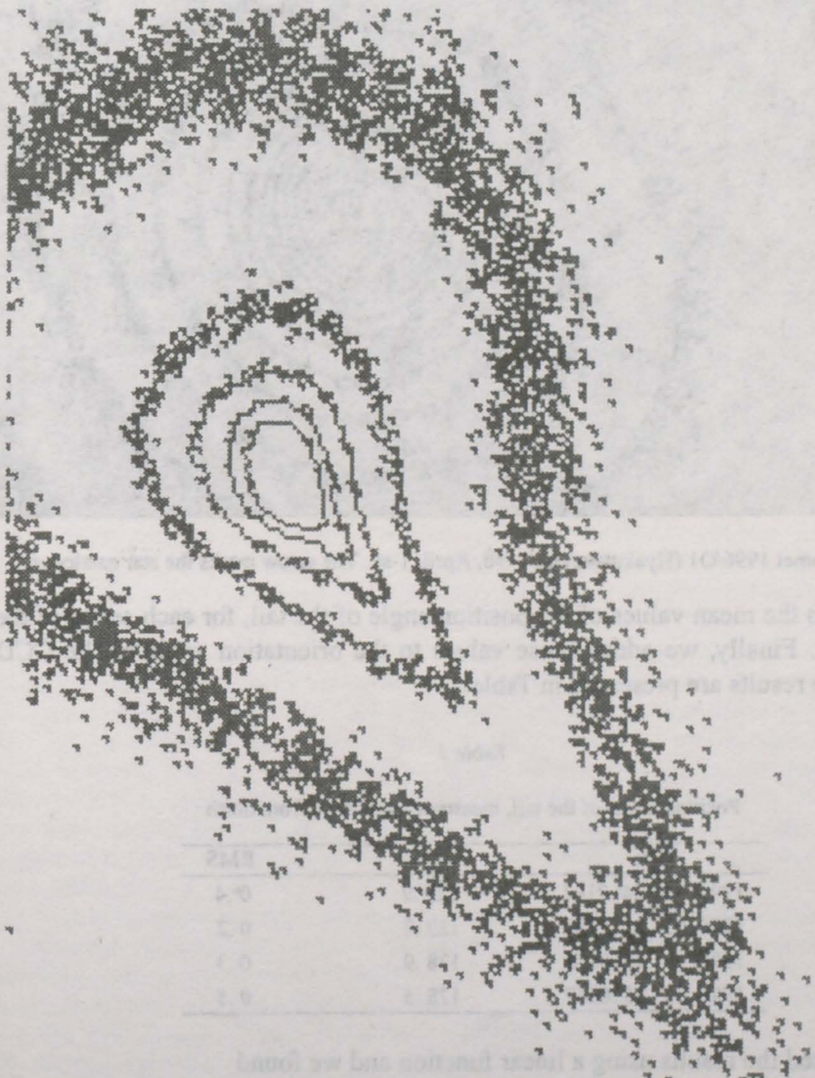

Fig. 4. – Comet C/1996B2 (Hyakutake) in 1996, March, 24-th.

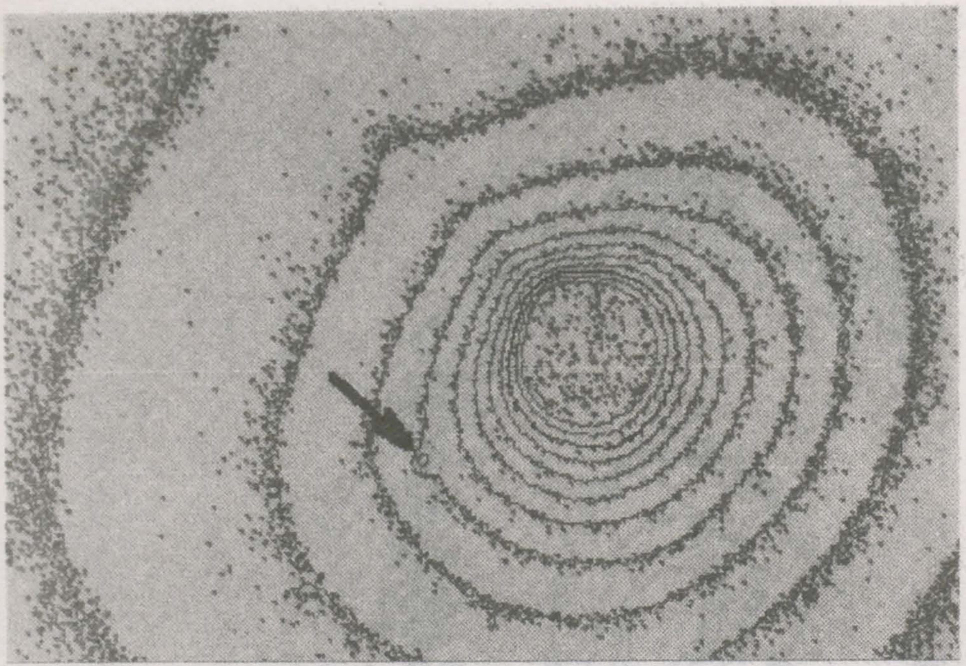

Fig. 5. – Comet 1996/O1 (Hyakutake) on 1996, April, 1-st . The arrow marks the star catalogue.

CCD system the mean values of the position angle of the tail, for each series of the five images. Finally, we added these values to the orientation angle of the CCD camera. The results are presented in Table 3.

Table 3

Position angle of the tail, measured clockwise from north

Date	Angle	RMS
$1996^y\ 03^m\ 24^d.9167$	$129^o.9$	$0^o.4$
1996 03 24 .9382	129 .7	0 .2
1996 03 25 .0243	128 .9	0 .3
1996 03 25 .0667	128 .5	0 .5

We fitted the results using a linear function and we found

$$\theta = -0.3886\, t + 138.45\ ,$$

where θ represents the orientation angle, and t is the time, given in hours.

Acknowledgements: We would like to thanks Dr. B. Marsden for promptly taking into consideration our astrometrical requests related to the comet. Special thanks Mrs. A. Alexiu for the "know-how" on the ASCORECORD measuring machine.

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Received on 14 January 1998

ANATOLE KATOK, BORIS HASSELBLATT, *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia of Mathematics and its Applications (ed. by G.-C. Rota), Vol. 54, Cambridge University Press, Cambridge, New York, Melbourne, 1995 (reprinted 1996, first paperback edition 1997), 802 pp., ISBN 0-521-34187-6 hardback, ISBN 0-521-57557-5 paperback.

The theory of dynamical systems is closely related to most of the main fields of mathematics. Although its area is circumscribed to the study of the global orbit structure of maps and flows, with special emphasis on properties invariant under coordinate changes, its concepts, methods, techniques and tools constitute a very efficient stimulus for research in many sciences. This book provides a self-contained coherent comprehensive exposition of the fundamentals of the theory of smooth dynamical systems together with related fields of other branches of dynamics: ergodic theory, symbolic dynamics, topological dynamics, etc. It is structured on four main parts.

Part 1 (*Examples and Fundamental Concepts*; Chapters 1-5) is intended to formulate the general program of the study of asymptotic properties. The main notions (e.g. differentiable and topological equivalence, structural stability, entropies, ergodicity, etc.) as well as certain basic methods (fixed point methods, coding, KAM-type Newton method, local normal forms, etc.) are introduced.

Part 2 (*Local Analysis and Orbit Growth*; Chapters 6-9) investigates the interplay between local analysis near individual orbits and the global complexity of the orbit structure via hyperbolicity, transversality, global topological invariants, and variational methods. Among such methods, the study of stable and unstable manifolds, bifurcations, index and degree, construction of orbits starting from action functionals are to be mentioned.

Parts 3 (*Low-Dimensional Phenomena*; Chapters 10-16) and 4 (*Hyperbolic Dynamical Systems*; Chapters 17-20) perform the program of analysis sketched in Part 1 to appreciable depth for two types of dynamical systems: low-dimensional and hyperbolic, very suitable for such an investigation. The complexity of hyperbolic systems is pointed out by the topological and statistical richness of orbit structure, and by stability under perturbation as well; these systems allow a very accurate, both quantitative and qualitative, description of their main characteristics. Some complexity is possible also for low-dimensional dynamical systems, although certain supplementary important aspects of the orbit structure can be understood only resorting to hyperbolicity or related kinds of behaviour.

The four main parts of the exposition are followed by a supplement (*Dynamical Systems with Nonuniformly Hyperbolic Behaviour*) by Anatole Katok and Leonardo Mendoza, intended to present the basic ideas, methods, and applications of the Pesin theory, and by an Appendix providing a background material for a better understanding of the book. In addition to the comments made in the text, the Notes groped after Appendix offer a broader survey of principal sources in the main areas of the theory of dynamical systems.

Throughout the book, many exercises (472) illustrate the use of results or methods presented in the text, or explore less discussed examples. Out of them, 317 are provided with hints or brief solutions in the back of the book.

The bibliography lists a great number of major monographs and representative textbooks and surveys covering the main branches of dynamics, landmark papers in the field, as well as many sources for the specific results presented along the text.

This book can be used both as a text for a course or for self-study and as a reference book. It is aimed at students and researchers in mathematics at all levels from advanced untergraduate up. Scientists and engineers working in applied dynamics, nonlinear science, and chaos will also find many fresh insights in this concrete and clear presentation.

Vasile Mioc

CHRONOBIOLOGY & ITS ROOTS IN THE COSMOS. Proceedings of the 3rd International Workshop organised by Slovak Medical Society, Institute of Preventive and Clinical Medicine, Research Institute of Rheumatic Diseases and Geophysical Institute of Slovak Academy of Sciences; held in The High Tatras, Slovakia, September 2-6, 1997; edited by Miroslav Mikulecký sen.

This workshop has been dedicated to Ladislav Dérer (1897-1960) and Svante Arrhenius (1859-1927). Four papers present these two personalities who have had a substantial contribution in the investigation of solar-terrestrial and of generally cosmic-terrestrial relations.

The main features of the workshop have been the interdisciplinary field and the collaboration spirit between researches from various specialities and different countries. The 290 pages containing 44 papers of the Proceedings are the result of the work - often collective - of mathematicians, physicists, geophysicians, astronomers, biologists, physicians from 16 countries. *A Few Words as a Welcome...* written by N. Marques (University of Sao Paolo, Brasilia), Head of the Scientific Committee of the Workshop, point up the main topic: the relationship between periodical astronomical, geophysical and biological events.

Dr. T. Trnovec specifies in the *Opening Address* the second aim of the workshop: to search for the mechanism of the possible interactions between cosmic factors and the terrestrial life. The main geophysical factors discussed at the section *Mediators of cosmo-terrestrial interactions* have been the geomagnetic field variability and the atmospheric pressure perturbations.

Several papers deal with the possible effects of the solar activity, considered as a *macroecological variable* (S. Erdel), in social, economic and medical processes. A paper tries to predict the sunspot number and a paper is devoted to the Sun as a complex physical system.

The coherence between the astronomical variations and the biorhythms is exposed in distinct sections, according to the period: a day (*Circadian rhythms in the framework of the chronome*), a week (*Dérer's circaseptans and Halberg's multiseptans*), a month (*Arrhenius' approach - lunar cycling?*). The papers about the seasonal variations, the multianual cycles (related to solar cycles) and the geographic latitude importance are presented together.

Despite the broad limits of the topics, the papers in this Proceedings have a common character: the use of reproducible data, processed by methods of time series analysis.

Irina Predeanu

JAROSLAV FOLTA (editor), *Mysterium Cosmographicum 1596-1996*, Proceedings of the Symposium held in Prague on 18-22 August 1996, Acta Historiae Rerum Naturalium necnon Technicarum, new series, Vol. 2, Prague, 1998, 316 p., ISSN 1211-958X.

Few scientists have had such a momentous contribution to the development of astronomy as Kepler. Dedicated to the 400th anniversary of the publication of Mysterium Cosmographicum, this volume of proceedings covers mainly topics concerning Kepler's work and life, but also problems of modern cosmology. Contributions dealing with the history of astronomy, proposing new theories with cosmologic or cosmogonic character, or tackling related problems, complete the volume.

Most of papers focus on Kepler's work. Domains as heliocentrism, planetary theory, cosmology, physical optics and acoustics, etc., to which Kepler has brought significant contributions, are surveyed, examined, and considered from new standpoints. In this way, new, surprising details are pointed out. Related aspects (life, relationships, correspondence, style of thinking, other works) also constitute a substantial part of the volume.

Another main topic regards the problems of the modern cosmology. In this context, subjects as non-Friedmannian models, distribution of the matter in the Universe, rotating black holes, etc., illustrate the link between the antique and the modern thinking as concerns birth and evolution.

Related to the above topics, there also are contributions dealing with the antique and the mediaeval history of astronomy. Among them, the overview of the development of astronomy in Romania deserves to be mentioned.

Papers with a rather special character are present, too: the comparison Kepler vs. Newton as regards the velocities of the celestial bodies; a model for the creation of the Universe; a theory concerning the explosive cosmogony of small bodies in the solar system.

The 32 contributions authored by 35 specialists from 14 countries provide a deep insight in Kepler's epoch, life, thinking, and work, as well as in the old and contemporary cosmological theories.

Magdalena Stavinschi

THE FOURTH YUGOSLAV-ROMANIAN ASTRONOMICAL MEETING

The fourth edition of the already traditional Yugoslav-Romanian Astronomical Meeting took place in Belgrade between 5–8 May 1998, coinciding with the 111-th anniversary of the Astronomical Observatory of Belgrade. The Romanian delegation, leaded by Dr Magda Stavinschi, Director of the Astronomical Institute of the Romanian Academy, was composed of nine researchers belonging to the Bucharest, Cluj-Napoca, and Timişoara Observatories of the Institute.

Set up in 1995, this cooperation in astronomy was organized as annual one-day round tables (Timişoara, July 1995; Belgrade, October 1996; Cluj-Napoca, September 1997). The success of these manifestations decided both parts to conceive the 1998 session as a several-day meeting with a printed volume of proceedings. This second step was successful, too.

The opening ceremony was honored by the presence of many outstanding personalities: the Minister of Foreign Affairs of Yugoslavia, the Minister of Development, Science, and Ecology of Yugoslavia, the minister of Science and Technology of Serbia, the President of the Serbian Academy of Sciences and Arts, the Ambassador of Romania at Belgrade, etc.

The conference has been structured on five sections: Astrophysics; Astrometry; Celestial Mechanics; Total Solar Eclipse on August 11, 1999; Astronomy in Archaeology, History, and Culture. Among them, the Romanian participants (M. Stavinschi, V. Mioc, G. Mariş) chaired three sessions. The contributions were presented as invited lectures, oral communications, and posters.

The invited lectures (among which we quote: M. S. Dimitrijević, *Belgrade Astronomical Observatory*; M. Stavinschi, *The Astronomical Observatory of Bucharest Has 90 Years*; Z. Knezević, *Advanced Theories to Compute Asteroid Mean Elements*; V. Mioc, *Specific Dynamical Features of Manev-Type Problems*; S. Ninković, *Models with Spherical Symmetry-Density Behaviour*) benefited of a special interest. As regards the communications and the posters, they gave the opportunity of animated and fruitful discussions. We must emphasize the occasion offered to the young researchers of both parts to present their results and to establish contacts with specialists working in the respective domains.

The round table dedicated to the total solar eclipse on August 11, 1999 was a hot point of the Conference. The importance (from all standpoints) of this event for Romania, the worldwide scientific (and not only) interest in it, the bilateral and multilateral cooperation in observing the phenomenon - all that was amply discussed. Scientific, educational, and public programs were established, and the necessity to involve all structures in was pointed out.

The Ambassador of Romania at Belgrade has received the Romanian delegation. This was a fruitful opportunity to discuss various and complex aspects of the Romanian-Yugoslav cooperation in the realm of science.

The exceptional efforts made by the Scientific Organizing Committee (co-chaired by Drs M.S. Dimitrijević and M. Stavinschi) and the Local Organizing Committee (chaired by Dr L.C. Popović) ensured a full success to the Conference. To give some examples, it is sufficient to mention: the large echo in mass media, the optimum conditions in carrying on, and the prompt apparition of the volume of Proceedings in excellent graphic conditions. All that constitutes a big challenge for the Romanian part in view of the next meeting, which will be organized in 1999 in Romania. We are sure that this will bring a supplementary contribution to the collaboration, friendship, and reciprocal relationship between the Romanian and Yugoslav astronomers.

Magdalena Stavinschi
Vasile Mioc

NOTICE TO AUTHORS

ROMANIAN ASTRONOMICAL JOURNAL is a journal which appears twice a year and is open to original contributions in Astronomy and related disciplines. The contributions – in English or France – can be accepted only if they were neither published before nor destined to any other publication.

Manuscrips should be submitted in duplicate; they must be typewritten on white A4-sized paper, onesided and doublespaced (enclosing abstract, references, footnotes and figure captions). The first page should contain: the article's title (brief and informative), author's name and affiliation, followed by an Abstract in English and Key words. The text should be clear an concise (it is recommended not to exceed 10 pages). *The Abstract* will present clearly the principal conclusions on the work, in no more than 10–15 lines.

Chapters an Paragraphs. Papers, except short notes, should be divided into Chapters, numbered by Arabic numerals. Chapters must be divided into Paragraphs denoted by the number of the Chapter and the number of the Paragraph; each chapter and each Paragraph should have a short descriptive title (eg. "3.2. Results").

Formulae have to be numbered consecutively in Arabic numerals, too, but included in parenthesis on the right side of the manuscript; all formulae should be written in legible form. The author should underline, in the text and in formulae, with lead pencil, all characters wich he wants to be italics; the bold appearance will be indicated by double underline.

Tables should be numbered consecutively in Arabic numerals; each should be typed on a separate sheet.

Figures and Illustrations should be submitted separately, in such a form as to permit reproduction without retouching. Any lettering should be large enough to be legible after the figure has been reduced in size for printing. Captions should be given on a separate sheet and labelled to show which illustration each is accompanying. All the figures should be numbered consecutively in Arabic numerals and reffered to in the text as e.g. Fig. 2 or Figs. 2–5. Photographs should be given only if essential and should be enlarged enough to permit clear reproduction.

The Places of tables and figures within the text have to be marked with lead pencil on the left margin of the manuscript.

References are indicated in the text by the author's name and year of publication. They should be listed in alphabetic and chronologic order at the end of the paper, as folows: name and the initials of the author(s), the year of publication, suitable abbreviation of the journal (or title of the book and editing house), its volume and page.

The content of the papers will be introduced on floppy disk (3.5"), in a well-known editor, preferably Word v.6. The collections and the emphasis will be made respecting (as far is posible) the prototype of the journal, 11/13 font for the text proper, 12/14 for the paper titles and 9/11 for annexes (tables, references, explanation, footnotes, etc.).

Please pay attention to these recommendations; it will contribute to a faster publishing of the manuscript.

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ROMANIAN
ASTRONOMICAL
JOURNAL

Vol. 8, No. 1, 1998

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ISSN 1210-5168

Rom. Astron. J., Vol. 8, No. 1, p. 1–58, Bucharest, 1998

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