[U]NIVERSITY OF BUCHAREST

Seminar

# TOPOLOGICAL ORDERED LINEAR SPACES
# NO. 18 (2001)

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# CONTENTS

## Papers

# A MATRICEAL ANALOGUE OF FEJER'S THEORY

SORINA BARZA, LARS-ERIK PERSSON, AND NICOLAE POPA

ABSTRACT. J. Arazy [A] pointed out that there is a similarity between functions defined on the torus $\mathbb{T}$ and the infinite matrices. In this paper we develop in the framework of matrices the theory Fejer developped for Fourier series.

## 0. Introduction

Let $A = (a_{ij})$, $i, j = 0, 1, 2, \ldots$ an infinite complex matrix.
For $k = 0, \pm 1, \pm 2 \ldots$, let us define $A_k = (a'_{ij})$, where

$$a'_{ij} = \begin{cases} a_{ij} & \text{if} \quad j - i = k \\ 0 & \text{otherwise.} \end{cases}$$

$A_k$ is called *the Fourier coefficient of k- order of the matrix A*. (See [Sh].) We have now a similarity between the expansion in the Fourier series $f = \sum_k a_k e^{ikx}$ of a periodical function $f$ on the torus $\mathbb{T}$ and the decomposition $A = \sum_{k \in Z} A_k$.

There is a similarity between the functions defined on the torus $\mathbb{T}$ and the infinite matrices, similarity remarked for the first time by Arazy [A] 1978 and exploited further by A. Shields [Sh] in 1983. Our main tool is an important characterization of Schur multipliers given by G. Bennett [B] in 1977.

Moreover, there is a similarity between the convolution product $f * g$ of two periodical functions and *the Schur product* of two matrices $A$ and $B$, $C = A * B$, where the matrix $C$ have the entries $c_{ij} = a_{ij} \cdot b_{ij}$, for $A = (a_{ij})_{i,j}$ and $B = (b_{ij})_{i,j}$. (See also [Sh].)

The aim of this paper is to extend in the framework of matrices the theory of Fejer developped for Fourier series. (See [H].)

In particular we mention the following results by Fejer, which have been guiding for our investigations:

(A) *A function* $f(\theta) = \sum_{k \in \mathbb{Z}} a_k e^{ik\theta}$ *on* $\mathbb{T}$ *is continuous on* $\mathbb{T}$, *that is* $f \in C(\mathbb{T})$, *if and only if the Cesaro sums* $\sigma_n(f) = \sum_{k=-n}^{n} a_k \left(1 - \frac{|k|}{n+1}\right) e^{ik\theta}$ *converge uniformly on* $\mathbb{T}$ *to* $f$.

(B) *A function* $f(\theta) = \sum_{k\in\mathbb{Z}} m_k e^{ik\theta} \in L^1$ *if and only if*

$$||\sigma_n(f) - f||_{L^1(\mathbb{T})} \xrightarrow[n\to\infty]{} 0.$$

The paper is organized in the following way: In order not to disturb our discussions later on we present some preliminaries in Section 1.

In Section 2, we derive some properties of and relations between the basic spaces $B(\ell_2)$ and $C(\ell_2)$ of independent interest.

The main results are presented in Section 3 and Section 4 is reserved for some concluding remarks and results.

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## 1. Preliminaries

In view of the Fejer's result (A) it is natural to give the following definition:

**Definition 1.** *Let* $A$ *be a matrix corresponding to an operator from* $B(\ell_2)$, *the space of all bounded operators on* $\ell_2$. *Denote now by* $\sigma_n(A)$ *the Cesaro sum associated to* $S_n(A) \stackrel{def}{=} \sum_{k=-n}^{n} A_k$, *that is* $\sigma_n(A) = \sum_{k=-n}^{n} A_k \left(1 - \frac{|k|}{n+1}\right)$.

*Then we say that* $A$ *is a continuous matrix if*

$$\lim_{n\to\infty} ||\sigma_n(A) - A||_{B(\ell_2)} = 0.$$

Let us denote by $C(\ell_2)$ *the vector space of all continuos matrices.*

On this space we introduce the following norm:

$$||A||_{C(\ell_2)} \stackrel{\text{def}}{=} \max[\sup_n ||\sigma_n(A)||_{B(\ell_2)}, ||A||_{B(\ell_2)}].$$

A matrix $M = (m_{ij})_{i,j}$ is called a *Schur multiplier* iff $M * A \in B(\ell_2)$ for all $A \in B(\ell_2)$.

The space of all Schur multipliers will be denoted by $\mathcal{M}(\ell_2)$ and the Schur multiplier norm of $M$ will be, by definition:

$$||M||_{\mathcal{M}(\ell_2)} \stackrel{\text{def}}{=} \sup_{||A||_{B(\ell_2)}\le 1} ||M * A||_{B(\ell_2)}.$$

Then it is known (see [B]) that $\mathcal{M}(\ell_2)$ *is a Banach space which is a commutative unital Banach algebra with respect to Schur product.*

Moreover, if $M$ is a Toeplitz matrix $M$, i. e. a matrix whose entries $m_{ij} = m_{j-i}$, for all $i, j \in \mathbb{N}^*$, then the following statement holds (see Thm. 8.1 - [B]).

(1) *The Toeplitz matrix $M$ is a multiplier if and only if there exists a bounded, complex, Borel measure $\mu \in \mathcal{M}(\mathbb{T})$, on $\mathbb{T}$ with the Fourier coefficients*

$$\hat{\mu}(n) = m_n \quad \text{for } n = 0, \pm 1 \pm 2, \dots$$

*Moreover, we then have*

$$||M||_{\mathcal{M}(\ell_2)} = ||\mu||_{\mathcal{M}(\mathbb{T})}.$$

We mention also the following well-known fact (see for instance [KZ])

*The Toeplitz matrix $M$ represents a linear and bounded operator on $\ell_2$ if and only if there exists a function $f \in L^\infty(\mathbb{T})$ with Fourier coefficients $\hat{f}(n) = m_n$ for all $n \in \mathbb{Z}$.*

*Moreover, we have*

$$||M||_{B(\ell_2)} = ||f||_{L^\infty(\mathbb{T})}.$$

## 2. Some properties of the space $C(\ell_2)$.

Let $C_\infty$ denote the space of all matrices defining compact operators.

**Proposition 1.** *$C(\ell_2)$ is a proper Banach subspace of $B(\ell_2)$ which, in its turn contains $C_\infty$ properly.*

*Proof.* It is clear that $||A||_{B(\ell_2)} \le ||A||_{C(\ell_2)}$ and, on the other hand, we have:

$$||\sigma_n(A)||_{B(\ell_2)} = || \sum_{k=-n}^{n} A_k \left(1 - \frac{|k|}{n+1}\right) ||_{B(\ell_2)} \le ||M_n||_{\mathcal{M}(\ell_2)} ||A||_{B(\ell_2)},$$

where $M_n$ is the matrix

$$\begin{pmatrix}
1 & 1-\frac{1}{n+1} & \dots & & 1-\frac{n}{n+1} & 0 & \dots \\
1-\frac{1}{n+1} & 1 & 1-\frac{1}{n+1} & & 1-\frac{n-1}{n+1} & 1-\frac{n}{n+1} & 0 \\
1-\frac{2}{n+1} & 1-\frac{1}{n+1} & 1 & \dots & 1-\frac{n-2}{n+1} & 1-\frac{n-1}{n+1} & 1-\frac{n}{n+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1-\frac{n}{n+1} & \dots & & 1-\frac{1}{n+1} & 1 & 1-\frac{1}{n+1} & \dots \\
0 & 1-\frac{n}{n+1} & & \dots & 1-\frac{1}{n+1} & 1 & 1-\frac{1}{n+1} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}.$$

Hence, by Theorem 8.1 -[B], we have:

$$||\sigma_n(A)||_{B(\ell_2)} \leq || \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ik\theta} ||_{\mathcal{M}(\mathbb{T})} \cdot ||A||_{B(\ell_2)}$$

$$= ||K_n(\theta)||_{\mathcal{M}(\mathbb{T})} \cdot ||A|| = ||A||_{B(\ell_2)}, \qquad (1)$$

since $K_n$, the Fejer kernel, has the $\mathcal{M}(\mathbb{T})$-norm equal to the $L^1(\mathbb{T})$-norm and finally equal to 1.

Hence

$$||A||_{C(\ell_2)} = ||A||_{B(\ell_2)}$$

and $C(\ell_2)$ is a normed subspace of $B(\ell_2)$.

But it is easy to see that $C(\ell_2)$ is a closed subspace of $B(\ell_2)$, that is $C(\ell_2)$ is a Banach subspace of $B(\ell_2)$. Next we note that $C(\ell_2)$ does not coincide with $B(\ell_2)$.

The matrix $A = \sum_{k\in\mathbb{Z}} A_k$, where $A_k = 0 \ \forall\ k < 0$ and $A_k = e_{k+1,2k+1}$, $k \geq 0$, $e_{ij}$ being the matrix whose single non-zero entry is 1 on the $i^{\text{th}}$ row and on the $j^{\text{th}}$ column, belongs to $B(\ell_2)$, since $(AA^*)^{1/2} = I$, the identity matrix. Moreover

$$||\sigma_n(A) - A||_{B(\ell_2)} = || \sum_{k>n} A_k + \frac{1}{n+1} \sum_{k=0}^{n} kA_k ||_{B(\ell_2)} = (\max_{k\leq n} \frac{k}{n+1}) \vee 1 = 1$$

for all $n$, thus $A \notin C(\ell_2)$.

Now let $A \in C_\infty$. Denoting by

$$P_n(A)(i,j) = \begin{cases} a_{ij} & i,j \leq n \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$||P_n(A) - A||_{B(\ell_2)} \to 0.$$

But, by Bennett's theorem, we have for $k > n$:

$$||P_n(A) - \sigma_k(P_n(A))||_{B(\ell_2)} = || \sum_{\ell=-n}^{n} (P_n(A))_\ell \frac{|\ell|}{k+1} ||_{B(\ell_2)} \leq$$

$$\leq || \sum_{\ell=-n}^{n} \frac{|\ell|}{k+1} e^{i\ell\theta} ||_{L^1(\mathbb{T})} \cdot ||P_n(A)||_{B(\ell_2)} \xrightarrow[k\to\infty]{} 0.$$

Hence $P_n(A) \in C(\ell_2)$ for all $n \in \mathbb{N}$, consequently $C_\infty \subset C(\ell_2)$.

Now let $A$ be the Toeplitz matrix given by $a(i,i+k)=\frac{1}{k^2}$ for all $i\in\mathbb{N}$ and $k\neq 0$. Then

$$||A||_{B(\ell_2)}=||\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{e^{ik\theta}}{k^2}||_{L^\infty(\mathbb{T})}\leq\sum_k\frac{1}{k^2}<\infty.$$

But it is well-known that a Toeplitz matrix does not represent a compact operator.

On the other hand

$$||\sum_{|k|\geq n}A_k+\frac{1}{n+1}\sum_{|k|\leq n}|k|A_k||_{B(\ell_2)}=||\sum_{|k|\geq n}\frac{e^{ik\theta}}{k^2}+\frac{1}{n+1}\sum_{|k|\leq n}\frac{e^{ik\theta}}{|k|}||_{L^\infty(\mathbb{T})}\leq$$

$$\leq C\left(\sum_{k\geq n}\frac{1}{k^2}+\frac{\log n}{n+1}\right),$$

which means that the Toeplitz matrix $A$ in fact belongs to $C(\ell_2)$ and we have also proved that $C_\infty$ is contained in $B(\ell_2)$ properly. ∎

**Proposition 2.** *$C(\ell_2)$ is a commutative Banach algebra whithout unit with respect to Schur multiplication.*

*Proof:* It suffices to observe that for $A,B\in C(\ell_2)$, $\sigma_n(A*B)=\sigma_n(A)*B$ and then we have for $A\in C(\ell_2)$, $B\in B(\ell_2)$

$$||A*B-\sigma_n(A*B)||_{B(\ell_2)}=||\,[A-\sigma_n(A)]*B||_{B(\ell_2)}\leq||A-\sigma_n(A)||_{B(\ell_2)}\cdot||B||_{\mathcal{M}(\ell_2)}$$

$$\leq||B||_{B(\ell_2)}\cdot||A-\sigma_n(A)||_{B(\ell_2)}.$$

Here we used the simple remark that

$$||B||_{\mathcal{M}(\ell_2)}=||B*\Delta||_{\mathcal{M}(\ell_2)}\leq||B||_{B(\ell_2)}\cdot||\Delta||_{\mathcal{M}(\ell_2)}=||B||_{B(\ell_2)},$$

where $\Delta_{ij}=1$ for all $i,j\in\mathbb{N}$, and $||\Delta||_{\mathcal{M}(\ell_2)}=1$. ∎

**Remark.** By Fejer's theorem (A) we have that a Toeplitz matrix $T=(t_k)_{k\in\mathbb{Z}}\in C(\ell_2)$ *if and only if* $f_T(\theta)\overset{def}{=}\sum_{k\in\mathbb{Z}}t_ke^{ik\theta}\in C(\mathbb{T})$, and in this way we can consider that the notion of a continuous matrix is the extension of that of continuous function.

## 3. The main results

Now we would like to give another characterization of the space $C(\ell_2)$, using the continuous vector-valued functions.

Consider now the function $f_A:\mathbb{T}\to B(\ell_2)$ given by $f_A(t)=\sum_{k\in\mathbb{Z}}A_ke^{ikt}$, then we ask ourselves: how should be the matrix $A$ in order that the function $f_A$ be a continuous $B(\ell_2)$-valued function.

The answer to this question is as follows:

**Theorem 1.** *Let $A$ be an infinite matrix.*

*Then $f_A$ is a $B(\ell_2)$-valued continuous function if and only if $A \in C(\ell_2)$, with equality of corresponding norms.*

*Proof.* $\Rightarrow$ Since $f_A(t) \in B(\ell_2)$ for all $t \in \mathbb{T}$, it follows that $A = f_A(0) \in B(\ell_2)$. The function $f_A : \mathbb{T} \to B(\ell_2)$ being continuous we can adapt the proof of Fejer's theorem (A) (see [H]-p.35) and we get that $\forall\ \delta > 0$ sufficiently small,

$$||\sigma_n(f_A) - f_A||_{C(\mathbb{T},B(\ell_2))} \leq \sup_{x\in\mathbb{T}} \sup_{|t|<\delta} ||f_A(x-t) - f_A(x)||_{B(\ell_2)} +$$

$$+2||f_A||_{C(\mathbb{T},B(\ell_2))} \cdot \sup_{|t|\geq\delta} K_n(t),$$

where $K_n(t)$ is the Fejer's kernel

$$K_n(t) = \frac{1}{n}\left[\frac{\sin\frac{n}{2}t}{\sin\frac{t}{2}}\right]^2, \quad t \in \mathbb{T} \text{ and } \sigma_n(f_A)(t) = \sum_{|k|\leq n}(1-\frac{|k|}{n+1})A_k e^{ikt} =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f_A(x)K_n(t-x)dx.$$

It follows that

$$\lim_n \sup_{t\in\mathbb{T}} ||\sigma_n(f_A)(t) - f_A||_{B(\ell_2)} = 0.$$

For $t = 0$, we get that $||\sigma_n(A) - A||_{B(\ell_2)} \underset{n}{\to} 0$, $\sigma_n(A)$ as in Section 1. Thus $A \in C(\ell_2)$.

Conversely, let $A \in C(\ell_2)$ and $f_A : \mathbb{T} \to B(\ell_2)$ as above. Then $\forall\ t \in \mathbb{T}$, we get

$$||\sigma_n(f_A)(t) - f_A(t)||_{B(\ell_2)} \leq ||\sigma_n(A) - A||_{B(\ell_2)} \cdot ||M(t)||_{\mathcal{M}(\ell_2)}$$

where $M(t)$ is the Toeplitz matrix with entries $(e^{ikt})_{k\in\mathbb{Z}}$.

But $||M(t)||_{\mathcal{M}(\ell_2)} = ||\sum_{k\in\mathbb{Z}} e^{ik(t+\theta)}||_{\mathcal{M}(\mathbb{T})} = ||\delta_{-t}||_{\mathcal{M}(\mathbb{T})} = 1,\ \forall\ t \in \mathbb{T}$.

Thus $\sup_{t\in\mathbb{T}}||\sigma_n(f_A)(t) - f_A(t)||_{B(\ell_2)} \leq ||\sigma_n(A) - A||_{B(\ell_2)}$, which in turn implies that $\sigma_n(f_A) \underset{n}{\to} f_A$ in the space $B(\mathbb{T}; B(\ell_2))$ of all bounded $B(\ell_2)$ -valued functions. Consequently, since $\sigma_n(f_A)$ are $B(\ell_2)$-valued continuous functions defined on $\mathbb{T}$, the same holds for $f_A$.

Moreover it is clear that $||A||_{C(\ell_2)} = ||f_A||_{C(\mathbb{T};B(\ell_2))}$. ∎

Now *what can you say about subspaces of $\mathcal{M}(\ell_2)$ in connection with multiplier property?*

The following theorem is a justification for introducing $C(\ell_2)$ and gives a partial answer to the above question. It is the matriceal analogue of Theorem 11.10 - Chap. IV - [Z], which precizes Theorem 8.1 -[B].

**Theorem 2.** *The Toeplitz matrix $M = (m_k)_{k\in\mathbb{Z}}$ is a Schur multiplier from $B(\ell_2)$ into $C(\ell_2)$ iff*

$$\sum_{k\in\mathbb{Z}} m_k e^{ik\theta} \in L^1(\mathbb{T}).$$

*Proof* If $A$ is Toeplitz matrix $A = (a_k)_{k\in\mathbb{Z}}$, then for any $\epsilon > 0$ there is $n_0 = n_0(\epsilon, A)$ such that for all $n \geq n_0$ and for all $p \in \mathbb{N}^*$, we have

$$||[\sigma_n(M) - \sigma_{n+p}(M)] * A||_{B(\ell_2)} \leq \epsilon.$$

(This inequality above means that $M * A \in C(\ell_2)$.)
But

$$|\frac{1}{2\pi}\int_0^{2\pi}[\sigma_n(\sum_k m_k e^{ik\theta}) - \sigma_{n+p}(\sum_k m_k e^{ik\theta})](t)(\sum_k a_k e^{-ikt})dt| \leq$$
$$\leq ||[\sigma_n(M) - \sigma_{n+p}(M)] * A||_{B(\ell_2)} \leq \epsilon \quad \forall n \geq n_0, \ \forall p > 0.$$

Consequently, taking $\sum_{k\in\mathbb{Z}} a_k e^{ikt} = \chi_E(t)$ for any measurable set $E \subset \mathbb{T}$, we have that

$$\int_E \sigma_n(\sum_k m_k e^{ik\theta})(t)dt$$

converges whenever $n \to \infty$, hence, by Theorem 9.13 (i)- Chap. IV - [Z] it follows that the functions
$\int_0^t \sigma_n(\sum_k m_k e^{ik\theta})(u)du$ are uniformly absolutely continuous. Thus

$$||\sigma_n(\sum_k m_k e^{ik\theta}) - \sigma_{n+p}(\sum_k m_k e^{ik\theta})||_{L^1} \xrightarrow[n,p\to\infty]{} 0.$$

Hence, by Fejer's theorem (B) it follows that $m(t) = \sum_k m_k e^{ikt} \in L^1(\mathbb{T})$.
Conversely, if $m(t) \in L^1(\mathbb{T})$ then

$$||\sigma_n(m(t)) - \sigma_{n+p}(m(t))||_{L^1(\mathbb{T})} \xrightarrow[n,p\to\infty]{} 0$$

hence

$$||\sigma_n(m(t)) - \sigma_{n+p}(m(t))||_{\mathcal{M}(\mathbb{T})} \xrightarrow[n,p\to\infty]{} 0,$$

which in turn implies

$$||\sigma_n(M * A) - \sigma_{n+p}(M * A)||_{B(\ell_2)} \leq ||\sigma_n(m) - \sigma_{n+p}(m)||_{\mathcal{M}(\mathbb{T})} \cdot ||A||_{B(\ell_2)},$$

for all $A \in B(\ell_2)$.

Consequently $M$ is a Schur multiplier from $B(\ell_2)$ into $C(\ell_2)$. ■

Now we define the notion of a *integrable matrix* in a similar way as we defined the notion of a continuous matrix, guided by (B).

**Definition 2.** *We say that an infinite matrix $A$ is an integrable matrix if $\sigma_n(A) \underset{n}{\rightarrow} A$ in the norm of $\mathcal{M}(\ell_2)$. The space of all such matrices, endowed with the norm induced by $\mathcal{M}(\ell_2)$, will be denoted by $L^1(\ell_2)$.*

Of course $L^1(\ell_2)$ is a Banach space.

**Remark.** *If $A \in L^1(\ell_2)$ then it follows that $A * B \in C(\ell_2)$ for all $B \in B(\ell_2)$.*

Indeed, for $B \in B(\ell_2)$, by

$$||\sigma_n(A * B) - A * B||_{B(\ell_2)} = ||\sigma_n(A) * B - A * B||_{B(\ell_2)} \leq$$
$$\leq ||\sigma_n(A) - A||_{\mathcal{M}(\ell_2)} \cdot ||B||_{B(\ell_2)} \xrightarrow[n\to\infty]{} 0,$$

we get $A * B \in C(\ell_2)$. ■

Now it is clear that $L^1(\ell_2)$ is a commutative Banach algebra (without unit) with respect to Schur product.

Indeed, for $A, B \in L^1(\ell_2)$ we have

$$||\sigma_n(A * B) - A * B||_{\mathcal{M}(\ell_2)} = ||[\sigma_n(A) - A] * B||_{\mathcal{M}(ell_2)} \leq$$
$$\leq ||\sigma_n(A) - A||_{\mathcal{M}(\ell_2)} \cdot ||B||_{\mathcal{M}(\ell_2)} \rightarrow 0,$$

which in turn implies that $A * B \in L^1(\ell_2)$.

View Theorem 2 and the remark above is natural to ask: *If $A$ is a Schur multiplier which maps $B(\ell_2)$ in $C(\ell_2)$ does it follow that $A \in L^1(\ell_2)$?*

The answer to the above question is negative:

**Example** *Let $A$ be the following matrix.*

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & & \vdots & \end{pmatrix}.$$

*Then it follows that $A$ is a Schur multiplier which maps $B(\ell_2)$ in $C(\ell_2)$ but does not belong to $L^1(\ell_2)$.*

$A$ is a Schur multiplier with the property that $A * B \in C(\ell_2)$ for all $B \in B(\ell_2)$ since the matrix $A * B$ has the rank 1, therefore represents a compact operator and consequently it belongs to $C(\ell_2)$.

$A$ does not belong to $L^1(\ell_2)$ by the lemma 1.

Therefore the Banach space $M(B(\ell_2), C(\ell_2))$ of all infinite matrices is different from both $\mathcal{M}(\ell_2)$ and $L^1(\ell_2)$.

It seems to us that this space deserves to be studied in more detail.

On the other hand the space $M(C(\ell_2), C(\ell_2))$ of all infinite matrices $A$ such that $A * B \in C(\ell_2)$ for all $B \in C(\ell_2)$ can be described easily.

More precisely we have:

**Theorem 3.** *$M(C(\ell_2), C(\ell_2))$ is exactly the space $\mathcal{M}(\ell_2)$ of all Schur multipliers.*

*Proof.* Since $\sigma_n(A * M) = \sigma_n(A) * M$ it follows easily that $M \in M(C(\ell_2), C(\ell_2))$ if $M \in \mathcal{M}(\ell_2)$ and $A \in C(\ell_2)$.

Conversely, let $M \in M(C(\ell_2), C(\ell_2))$. For $A \in C(\ell_2)$ the map $A \to M * A$ from $C(\ell_2)$ into $C(\ell_2)$ has clearly a closed graph. Now let $M * A_n \to B$ in $B(\ell_2)$. Then obviously $B = M * A$. Therefore by closed graph theorem it follows that $A \to M * A$ is a continuous map from $C(\ell_2)$ into $C(\ell_2)$, that is there is $c > 0$ such that $||M * A|| \leq c \cdot ||A||_{B(\ell_2)}$ for all $A \in C(\ell_2)$.

By the definition it follows easily that

$$\sup_n ||P_n A||_{B(\ell_2)} = ||A||_{B(\ell_2)}$$

where

$$P_n A(i,j) = \begin{cases} A(i,j) & i \leq n \\ 0 & \text{otherwise.} \end{cases}$$

But $P_n A \in C(\ell_2)$ and therefore

$$||M * P_n A||_{B(\ell_2)} \leq c||P_n A||_{B(\ell_2)} \leq c||A|| \quad \forall A \in B(\ell_2), \forall n.$$

Thus $||P_n(M * A)||_{B(\ell_2)} \leq c||A||_{B(\ell_2)}$ for all $n$. Consequently $||M * A||_{B(\ell_2)} \leq c||A||_{B(\ell_2)}$ for all $A \in B(\ell_2)$ that is $M$ is a Schur multiplier. ∎

**Proposition 3.** *If $M \in L^1(\ell_2)$ and $\epsilon > 0$ there is $M_1 \in C(\ell_2)$ and $M_2 \in L^1(\ell_2)$ such that $M = M_1 + M_2$, where $||M_2||_{L^1(\ell_2)} \leq \epsilon$.*

(The above Proposition is the matriceal analogue of Luzin's theorem.)

*Proof* By Definition 1 there is an $n$ such that $M_2 \stackrel{\text{def}}{=} M - \sigma_n(M)$ verifies $||M_2||_{L^1(\ell_2)} \leq \epsilon$.

But obviously $M_1 = \sigma_n(M) \in C(\ell_2)$ (as a finite sum of diagonals $M_k$.) ∎

Now we get the following analogue of Riemann-Lebesgue Lemma:

**Lemma 1.** *Let $M \in L^1(\ell_2)$. Then*

$$\lim_{|k| \to \infty} ||M_k||_{L^1(\ell_2)} = 0.$$

*Proof* We use the decomposition given by the above proposition and we get, for $\epsilon > 0$, that, if $|k| \geq n(\epsilon)$, then $(M_1)_k = 0$, $(M_2)_k = M_k$ and $||(M_2)_k||_{L^1(\ell_2)} \leq ||M_2||_{L^1(\ell_2)} \leq \epsilon$.

Hence for all $\epsilon > 0$ there is an $n(\epsilon)$ such that for $|k| \geq n(\epsilon)$ it follows that $||M_k||_{L^1(\ell_2)} \leq \epsilon$. ∎

We give now a characterization of integrable matrices in the spirit of Theorem 1:

We recall that by $L^1(\mathbb{T}, \mathcal{M}(\ell_2))$ we mean the space of all Bochner integrable $\mathcal{M}(\ell_2)$-functions with the norm $||f|| = \int_{\mathbb{T}} ||f(t)||_{\mathcal{M}(\ell_2)} dt$.

**Theorem 4.** *The matrix $A \in L^1(\ell_2)$ if and only if the function $f_A(t) \in L^1(\mathbb{T}, \mathcal{M}(\ell_2))$. Moreover the norms of both spaces are equivalent.*

*Proof.* Let $A \in L^1(\ell_2)$. Then it follows that $\lim_{n\to\infty} ||\sigma_n(A) - A||_{\mathcal{M}(\ell_2)} = 0$. We consider now the function $f_A$ and we have the relation:

$$||\sigma_n(f_A)(t) - f_A(t)||_{\mathcal{M}(\ell_2)} \leq ||\sigma_n(A) - A||_{\mathcal{M}(\ell_2)} \cdot ||(e^{ikt})_{k\in\mathbb{Z}}||_{\mathcal{M}(\ell_2)} =$$

$$= ||\sigma_n(A) - A||_{\mathcal{M}(\ell_2)},$$

which in turn implies

$$\lim_{n\to\infty} ||\sigma_n(f_A) - f_A||_{L^1(\mathbb{T},\mathcal{M}(\ell_2))} \leq \lim_{n\to\infty} ||\sigma_n(A) - A||_{\mathcal{M}(\ell_2)} = 0,$$

that is the Cesaro sums $\sigma_n(f_A)$ associated to the function $f_A$ converge in $L^1(\mathbb{T}, \mathcal{M}(\ell_2))$.

But this implies that $f_A$ belongs to $L^1(\mathbb{T}, \mathcal{M}(\ell_2))$.

Conversely,

$$||\sigma_n(f_A) - f_A||_{L^1(\mathbb{T},\mathcal{M}(\ell_2))} = \frac{1}{2\pi} \int_{\mathbb{T}} ||\frac{1}{2\pi} \int_{\mathbb{T}} [f_A(t-\theta) - f_A(t)] K_n(\theta) d\theta||_{\mathcal{M}(\ell_2)} dt \leq$$

$$\leq \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1}{2\pi} \int_{\mathbb{T}} ||f_A(t-\theta) - f_A(t)||_{\mathcal{M}(\ell_2)} K_n(\theta) d\theta dt = (\text{ by Fubini' s theorem })$$

$$= \frac{1}{2\pi} \int_{\mathbb{T}} K_n(\theta) ||(f_A)_\theta - f_A||_{L^1(\mathbb{T},\mathcal{M}(\ell_2))} d\theta \leq \sup_{|\theta|<\delta} ||(f_A)_\theta - f_A||_{L^1(\mathbb{T},\mathcal{M}(\ell_2))} +$$

$$+ 2||f_A||_{L^1(\mathbb{T},\mathcal{M}(\ell_2))} \cdot \sup_{|\theta|\geq\delta} K_n(t),$$

where $(f_A)_\theta(t) \stackrel{\text{def}}{=} f_A(t-\theta)$.

But, for $\delta$ sufficiently small it follows that

$$\int_{\mathbb{T}} ||f_A(t-\theta) - f_A(t)||_{\mathcal{M}(\ell_2)} dt < \epsilon$$

On the other side we have

$$\lim_{n\to\infty} \sup_{|t|\geq\delta} K_n(t) = 0,$$

therefore it follows that

$$\lim_{n\to\infty} ||\sigma_n(f_A) - f_A||_{L^1(\mathbb{T},\mathcal{M}(\ell_2))} = 0.$$

But we may find a subsequence $\sigma_{n_k}(f_A)$ which converges a.e. on $\mathbb{T}$ to $f_A$. Then, it is some $t_0 \in \mathbb{T}$ such that

$$||\sigma_{n_k}(A) - A||_{\mathcal{M}(\ell_2)} \leq ||\sigma_{n_k}(f_A)(t_0) - f_A(t_0)||_{\mathcal{M}(\ell_2)} \cdot ||(e^{-ikt_0})_k||_{\mathcal{M}(\mathbb{T})} =$$

$$= ||\sigma_{n_k}(f_A)(t_0) - f_A(t_0)||_{\mathcal{M}(\ell_2)} \xrightarrow[k\to\infty]{} 0.$$

$L^1$ being a closed subspace in $\mathcal{M}(\ell_2)$, it follows that $A \in L^1(\ell_2)$.

The equivalence of the norms follows from the obvious inequality $||f_A||_{L^1(\mathbb{T},\mathcal{M}(\ell_2))} \leq ||A||_{\mathcal{M}(\ell_2)}$ and by Banach's isomorphisms theorem. ∎

**Remark.** *Let $A$ be a Toeplitz matrix. Then $A \in L^1(\ell_2)$ if and only if $f_A$ belongs to the subspace of $L^1(\mathbb{T}, \mathcal{M}(\ell_2))$ consisting of all Toeplitz matrices, therefore belongs to a space isomorphic to $L^1(\mathbb{T})$.*

## 4. Concluding remarks and results.

We recall the following well-known result (see Theorem in Chapter 2 of [H]):

(2) *A function $f$ on $\mathbb{T}$ belongs to $L^\infty(\mathbb{T})$ if and only if*

$$\sup_n ||\sigma_n(f)||_{L^\infty(\mathbb{T})} < \infty.$$

We have the following matriceal analogue of (2):

**Proposition 4.** *Let $A$ be an infinite matrix. Then $A$ belongs to $B(\ell_2)$ iff*

$$\sup_n ||\sigma_n(A)||_{B(\ell_2)} < \infty.$$

*Proof* Let $\sup_n ||\sigma_n(A)||_{B(\ell_2)} < \infty$. Then, by Alaoglu theorem, since, for all $i, j \in \mathbb{N}$, we have $\sigma_n(A)(i,j) \xrightarrow[n\to\infty]{} a_{i,j}$, it follows that $A \in B(\ell_2)$.

The converse holds by the proof of Proposition 1. ∎

**Proposition 5.** $A \in \mathcal{M}(\ell_2)$ *if and only if*

$$\sup_n ||\sigma_n(A)||_{\mathcal{M}(\ell_2)} < \infty.$$

*Proof.* We use the well-known fact that the space $\mathcal{M}(\ell_2)$ of all Schur multipliers is a Banach dual space , namely $\mathcal{M}(\ell_2)$ is the dual space of $\ell_1 \otimes_\alpha \ell_1$, where

$$\alpha(v) = \inf\{|\sum_i |\beta_i|^2)^{1/2}(\sum_j |\alpha_j|^2)^{1/2}||(a_{ij})||_{B(\ell_2,\ell_2)}\}$$

where the infimum runs over all representations of $v$ of the form $\sum_{i,j\in\mathbb{N}^*} \alpha_j a_{ij}\beta_i e_i \otimes e_j$. (see [P]). Then by Alaoglu's theorem we get that if

$$\sup_n ||\sigma_n(A)||_{\mathcal{M}(\ell_2)} < \infty$$

then $A \in \mathcal{M}(\ell_2)$.

Conversely, let $A \in \mathcal{M}(\ell_2)$. Then $\sigma_n(A) = A * \sigma_n(M)$, where $M = (m_i)_{i\in\mathbb{Z}}$ with $m_i = 1$.

Thus

$$||\sigma_n(A)||_{\mathcal{M}(\ell_2)} \le ||A||_{\mathcal{M}(\ell_2)} \cdot ||\sigma_n(M)||_{\mathcal{M}(\ell_2)} \le$$
$$\le \text{(by Theorem 8.1 -[B])} \le ||A||_{\mathcal{M}(\ell_2)}. \blacksquare$$

Another characterization of matrices from $\mathcal{M}(\ell_2)$ is as follows: First we attach to each infinite matrix $A$ a linear bounded operator $\mu_A : L^1(\mathbb{T}) \to \mathcal{M}(\ell_2)$, given by $\mu_A(g) = A * G$, where $g \in L^1(\mathbb{T})$ and $G$ is the Toeplitz matrix corresponding to $g$, matrix belonging to $L^1(\ell_2)$.

More specifically we have the following result:

**Theorem 5.** *Let $A$ be an infinite matrix. Then $A \in \mathcal{M}(\ell_2)$ if and only if $\mu_A \in L(L^1(\mathbb{T}), \mathcal{M}(\ell_2))$, the corresponding norms being equivalent.*

*Proof.* Let $A \in \mathcal{M}(\ell_2)$. We consider the linear operator $\mu_A : L^1(\mathbb{T}, \mathcal{M}(\ell_2))$, given by $\mu_A(g) = A * G$, where $G$ is as above.

$\mu_A$ is a bounded operator. Indeed

$$||\mu_A(g)||_{\mathcal{M}(\ell_2)} = ||A*G||_{\mathcal{M}(\ell_2)} \le ||A||_{\mathcal{M}(\ell_2)} \cdot ||G||_{L^1(\ell_2)} = ||A||_{\mathcal{M}(\ell_2)} \cdot ||g||_{L^1(\mathbb{T})},$$

$\forall g \in L^1(\mathbb{T})$, which, in turn, implies that $||\mu_A||_{L(L^1(\mathbb{T}),\mathcal{M}(\ell_2))} \le ||A||_{\mathcal{M}(\ell_2)}$.

Conversely, let $\mu_A \in L(L^1(\mathbb{T}), \mathcal{M}(\ell_2))$, $g \in L^1(\mathbb{T})$. Then, using the above notations, we get:

$$||\sigma_n(A)||_{\mathcal{M}(\ell_2)} = ||A * K_n||_{\mathcal{M}(\ell_2)} = ||\mu_A(k_n)||_{\mathcal{M}(\ell_2)} \le ||\mu_A||_{L(L^1(\mathbb{T}),\mathcal{M}(\ell_2))}$$

for all $n \in \mathbb{N}$, since the Fejer's kernels $k_n$ have the $L^1$-norms equal to one, for all $n \in \mathbb{N}$. Then, by Proposition 5 it follows that $A \in \mathcal{M}(\ell_2)$ and $||A||_{\mathcal{M}(\ell_2)} \le C||\mu_A||_{L(L^1(\mathbb{T}),\mathcal{M}(\ell_2))}. \blacksquare$

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# ON THE DERIVATION OF SET'S FUNCTIONS

by Ileana Bucur

We consider a metric space $(X,d)$. As usually we denote:

$$B(x,r):=\{y\in X \mid d(y,x)<r\} \quad \forall\ x\in X,\ \forall\ r>0.$$

$$\delta(A):=\sup\{d(x,y) \mid x,y\in A\} \quad \forall\ A\subset X, A\neq\emptyset.$$

$\mathcal{B}=\mathcal{B}(X)$ – the class of all Borel sets of $X$ i.e. the σ-algebra generated by the family of all open subsets of $X$.

**Definition 1.** *A positive measure $\lambda:\mathcal{B}(X)\to\overline{\mathbb{R}}_+$ is called a **Vitali-measure** if*

*a)* $\quad 0<\lambda(B(x,r))<\infty \qquad \forall\ x\in X,\ \forall\ r>0.$

*b) there exist a positive number θ such that*

$$\lambda(B(x,2r))\le\theta\lambda(B(x,r)) \quad \forall\ x\in X,\ \forall\ r>0.$$

*For any nonempty set $A\in B$ we denote*

$$\alpha(A):=\sup\left\{\frac{\lambda(A)}{\lambda(B(x,r))} \,\Big|\, A\subset B(x,r)\right\}$$

*and a sequence $(A_n)_n$ of Borel subsets of X will be termed **regular** if*

$$\inf\{\alpha(A_n) \mid n\in\mathbb{N}\}>0.$$

**From now on we suppose that λ is fixed Vitali measure on $(X,d)$.**

**Definition 2.** *A sequence $(F_n)_n$ of closed subsets of X is called **convergent to** $x_0\in X$ if $x_0\in F_n$ for any $n\in\mathbb{N}$ and $\lim_{n\to\infty}\delta(F_n)=0$.*

*Let E be a Banach space. A map $\mu:\mathcal{B}\to E$ is called: **additive** if $\mu(A_1\cup A_2)=$ $=\mu(A_1)+\mu(A_2)$ for all paire $(A_1,A_2)$ of Borel, disjoint sets of X.*

**Definition 3.** *An additive map $\mu:\mathcal{B}\to E$ is termed **regular** at a point $A\in\mathcal{B}$ if for any positive number ε, ε ≠ 0, a closed subset F and an open subset G of X such that $F\subset A\subset G$ and for any finite $\mathcal{B}$-partition $(B_i)_{i\le k}$ of $G\setminus F$ (i.e. $B_i\in\mathcal{B}\ \forall\ i\le k$, $B_i\cap B_j=\emptyset$ if $i\neq j$, $G\setminus F=\bigcup_{i\le k}B_i$) we have*

$$\sum_{i\le k}\|\mu(B_i)\|\le\varepsilon.$$

*The additive map $\mu:\mathcal{B}\to E$ will be called **regular on a subset** $\mathcal{M}$ of $\mathcal{B}$ if μ is regular at any point $A\in\mathcal{M}$.*

*We recall also that μ has **bounded variation** on a Borel set A of X if there exist a positive number M such that*

$$\sum_{n\le k}\|\mu(A_n)\|\le M$$

*for all finite $\mathcal{B}$-partition $(A_i)_{i\le k}$ of A.*

It is almost obvious the following assertion.

**Proposition 1.** ***If** $\mu:\mathcal{B}\to E$ **is additive then we have:***

***a) the map** $:|\mu|:\mathcal{B}\to\overline{\mathbb{R}}_+$ **-termed the variation of** μ*

$$|\mu|(A)=\sup_{(A_i)_i}\sum_{i\le l}\|\mu(A_i)\|,$$

***where*** $(A_i)_{i\le k}$ ***runs the family of all finite*** $\mathscr{B}$***-partition of A, is additive.***

*b)* $\mu$ ***has bounded variation on*** $A\in\mathscr{B}$ ***iff*** $|\mu|(A)<\infty$.

*c)* $\mu$ ***is regular at a point*** $A\in\mathscr{B}$ ***iff*** $|\mu|$ ***is regular at this point.***

*d) if* $A\in\mathscr{B}$ ***the restriction of*** $\mu$ ***to*** $A$ *i.e. the map* $\mu_A:\mathscr{B}\to E$, $\mu_A(M)=\mu(A\cap M)$ $\forall\ M\in\mathscr{B}$ ***is also additive; moreover*** $\mu_A$ ***has bounded variation on X iff*** $\mu$ ***has bounded variation on*** $A$ ***and*** $\mu$ ***is regular at a point*** $A'\in\mathscr{B}$, $A'\subset A$ ***iff*** $\mu_A$ ***is regular at*** $A'$.

*e)* ***if*** $A\in\mathscr{B}$ ***and*** $B=X\setminus A$ ***then*** $\mu=\mu_A+\mu_B$, $|\mu|=|\mu_A|+|\mu_B|$.

**Definition 4.** *The additive map* $\mu:\mathscr{B}\to E$ *is called* ***derivable at a point*** $x_0\in X$ *if for all regular sequence* $(F_n)_n$ *of closed subsets of X, converging to* $x_0$, *the sequence*

$$\left(\frac{\mu(F_n)}{\lambda(F_n)}\right)_n$$

*converges in E.*

*We remember two assertions which generalize in some sens the well known Lebesque theorem on derivation.*

**Theorem 2.** ***If*** $\mu:\mathscr{B}\to\mathbb{R}_+$ ***is a positive measure then*** $\mu$ ***is derivable on X outside a*** $\lambda$***-negligible set*** *(or equivalently* $\lambda$ *a.e. on X).*

For the details on the proof one can see [M.N], 25.30.

**Theorem 3.** ***If*** $\mu:\mathscr{B}\to\mathbb{R}$ ***is a bounded additive and regular map on*** $\mathscr{B}$ ***then*** $\mu$ ***is derivable*** $\lambda$ ***a.e. on X.***

For more details one can see [I.B-D].

The aim of this paper is to extend Theorem 2 to the case where $\mu$ is and additive vector valued map namely:

**Theorem 4.** ***Let E be a Bannach space,*** $\mu:\mathscr{B}\to E$ ***be an additive map which is regular and has finite variation on any bounded subset A of*** $\mathscr{B}$***.***

***If there exist an increasing sequence*** $(X_n)_n$ ***in*** $\mathscr{B}$ ***such that***

$$\lambda\left(X\setminus\bigcup_{n=1}^{\infty}X_n\right)=0,$$

$$E_n:=\left\{\frac{\mu(A)}{\gamma(A)}\,\Big|\,A\in B,\ A\subset X_n,\ 0<\gamma(A)\right\}\ \textbf{\textit{is compact}}\ \forall\ n\in\mathbb{N}^*$$

***then*** $\mu$ ***is derivable*** $\lambda$ ***a.e. on X.***

***Proof.*** First we show the following assertions:

a) **If** $\theta:\mathscr{B}\to E$ **is „carred" by a closed subset** $F$ **of** $X$**, i.e.** $\theta(A)=0$ **for all** $A\in\mathscr{B}$, $A\cap F=\varnothing$ **then** $\theta$ **is derivable at any point** $x\in X\setminus F$**.**

**b) If** $\theta:\mathscr{B}\to E$ **has bounded variation, is carred by a set** $A_0\in\mathscr{B}$ **(**$\theta(A')=0$ **if** $A'\cap A_0=\varnothing$**) and** $\theta$ **is regular at** $A_0$ **then** $\theta$ **is derivable** $\lambda$ **a.e. on** $X\setminus A_0$**.**

Indeed, the assertion a) follows from the fact that for any point $x_0\notin F$ there exist $r>0$ such that $\overline{B}(x_0,r)\cap F=\varnothing$. In this case, if we consider a regular sequence $(F_n)_n$ of closed subsets converging to $x_0$ we may choose $n_0\in\mathbb{N}$ such that

$$n \geq n_0 \Rightarrow F_n \subset \overline{B}(x_0, r), \quad \frac{\theta(F_n)}{\lambda(F_n)} = 0.$$

Hence $\theta$ is derivable at $x_0$ and the derivative $\theta'(x_0)$ of $\theta$ at $x_0$ is equal 0.

b) We show that outside a $\lambda$-negligible subset we have $\theta'(x) = 0$ on $X \setminus A_0$. For this it will be sufficient to show that for any $\rho > 0$ and any $\varepsilon > 0$ we have $\lambda^*(M_\rho) < \varepsilon$ where $M_\rho$ is the set of all point $x \in X \setminus A_0$ for which there exist a regular sequence $(F_n^x)_n$ of closed subsets converging to $x$ such that

$$\lim_{n \to \infty} \frac{|\theta|(F_n^x)}{\lambda(F_n^x)} > \rho.$$

So we fix $\varepsilon > 0$ and we consider a closed subset $F_\varepsilon$ of $A_0$ such that

$$|\theta|(A_0 \setminus F_\varepsilon) < \varepsilon.$$

Then we decompose $\theta$ under the form $\theta = \theta_1 + \theta_2$ where $\theta_1, \theta_2$ are defined by

$$\theta_1(A) = \theta(F_\varepsilon \cap A), \quad \theta_2(A) = \theta(A \cap (A_0 \setminus F_\varepsilon))$$

From the preceding point a) we have $|\theta_1|'(x) = 0$ at any point $x \in X \setminus F_\varepsilon$ since $|\theta_1|$ is carred by $F_\varepsilon$.

Hence $|\theta_1|'(x) = 0$ at any point $x \in X \setminus A_0$.

If $x \in M_\rho$ there exist a regular sequence $(F_n^x)_n$ of closed subsets of $X$, converging to $x$, $F_n^x \subset X \setminus F_\varepsilon$ such that

$$\rho < \lim_{n \to \infty} \frac{|\theta|(F_n^x)}{\lambda(F_n^x)} = \lim_{n \to \infty} \frac{|\theta_2|(F_n^x)}{\lambda(F_n^x)}.$$

Using Vitali covering lemma we may choose a countable family $(F_n)_n$ of pairwise closed subset, $F_n \subset X \setminus F_\varepsilon$ such that

$$\frac{|\theta_2|(F_n)}{\lambda(F_n)} > \rho \quad \forall\ n \in \mathbb{N}, \quad \lambda^*\left(M_\rho \setminus \bigcup_{n \in \mathbb{N}} F_n\right) = 0.$$

We have

$$\rho \sum_{n \in \mathbb{N}} \lambda(F_n) \leq \sum_{n \in \mathbb{N}} |\theta_2|(F_n) \leq |\theta_2|(X \setminus F_\varepsilon) = |\theta_2|(A \setminus F_\varepsilon) < \varepsilon,$$

$$\lambda^*(M_\rho) = \lambda\left(\bigcup_n F_n\right) \leq \varepsilon/\rho.$$

Let now $(X_n)_n$ be an increasing sequence of $\mathscr{B}$ such that $\lambda\left(X \setminus \bigcup_n X_n\right) = 0$ such that the set

$$E_n = \left\{\frac{\mu(A)}{\lambda(A)} \middle| A \in \mathscr{B}, \lambda(A) > 0, A \subset X_n\right\}$$

is relatively compact. We replace any $X_n$ by the set $X_n'$ defined by

$$X_n' = X_n \cap \mathscr{B}(x_0, n) \quad \forall\ n \in \mathbb{N}^*$$

where $x_0$ is a fixed point in $X$. Obviously the sets

$$E_n' := \left\{ \frac{\mu(A)}{\lambda(A)} \middle| A \in \mathscr{B}, \lambda(A) > 0, A \subset X_n' \right\}, n \in \mathbb{N}^*$$

are relatively compact. For any $n$ we consider the additive maps $\mu_n, \theta_n, \gamma_n$ defined on $\mathscr{B}$ as follows

$$\mu_n(A) = \mu(A \cap X_n'),\ \theta_n(A) = \mu\big(A \cap (X \setminus X_n') \cap B(x_0, n)\big),$$
$$\gamma_n(A) = \mu\Big(A \cap (X \setminus X_n') \cap \big(X \setminus B(x_0, n)\big)\Big).$$

Since $\gamma_n$ is carred by the closed set $X \setminus B(x_0, n)$ and $X_n' \subset B(x_0, n)$, using assertion a) we deduce that $\gamma_n$ is derivable at any point $x \in X_n'$ and $\gamma_n'(x) = 0$. Since $\theta_n$ is carred by the bounded set $B(x_0, n) \cap (X \setminus X_n')$ we deduce that $\theta_n$ is with bounded variation and regular. Hence, using assertion b) we deduce that $\theta_n$ is derivable at any point of $X_n'$ outside a $\lambda$-negligible subset $A_n \subset X_n'$.

Since the set $E_n'$ is relatively compact, i.e. the closure $K_n$ of $E_n'$ is compact subset of $E$ then the topology of $K_n$ coincides with the $\sigma(K_n, E')$-topology. Hence we may consider a sequence $(f_m)_m$ in $E'$ which separates the points of $K_n$ and therefore a sequence $(x_p)_p$ of $K_n$ is convergent iff for any $m \in \mathbb{N}$ the real sequence $\big(f_m(x_p)\big)_p$ is convergent.

Now for any $m \in \mathbb{N}$ the map $\omega_m$ on $\mathscr{B}$ with values in $\mathbb{R}$ defined by

$$A \to f_m\big(\mu_n(A)\big) = \omega_m(A)$$

is regular, with bounded variation and additive. From Theorem 3 we deduce that $\omega_m$ is derivable on $X$ outside a $\lambda$-negligible set $B_m^n \in \mathscr{B}$.

Hence outside the $\lambda$-negligible set $\bigcup_{m \in \mathbb{N}} B_m$ all the maps $\omega_m$ are derivable on $X$ and therefore the map $\mu_n$ is derivable on $X_n' \setminus \bigcup_m B_m^n$. Using all preceding considerations we deduce that $\mu$ is derivable on $X_n'$ outside the $\lambda$-negligible subset $A_n \cup \bigcup_m B_m^n$. Hence outside the $\lambda$-negligible subset of $X$ given by

$$\bigcup_n \left( A_n \cup \bigcup_m B_m^n \right) \cup \left( X \setminus \bigcup_n X_n' \right)$$

the map $\mu$ is derivable.

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# Absolute continuity and Radon-Nikodým representations into functional framework

Ion Chiţescu

## I. General setting

We shall consider a *Loomis system* $(X, \mathcal{B}, I)$. Namely, $X$ is a non empty set, $\mathcal{B}$ is a vector lattice of functions $f : X \to I\!R$ (pointwise order and operations) and $I : \mathcal{B} \to I\!R$ is a linear and positive functional. Write $\mathcal{B}_+ := \{u \in \mathcal{B} | u \geq 0\}$. We shall also assume that $\mathcal{B}$ is an algebra with unit (i.e. for all $u, v$ in $\mathcal{B}$ one has $uv \in \mathcal{B}$ and the constant function $1 \in \mathcal{B}$). In the particular case when $I(u_n) \to 0$ for every decreasing sequence $u_n \downarrow 0$ (pointwise), the functional $I$ is called a Daniell integral. We shall also consider another linear and positive functional $J : \mathcal{B} \to I\!R$.

**Definition.** *We say that $J$ is absolutely continuous with respect to $I$ (and we shall write $J \ll I$) if for all $\varepsilon > 0$ and for all $h \in \mathcal{B}_+$, there exists $\delta > 0$ such that for all $\mathcal{B}_+ \ni u \leq h$ with $I(u) < \delta$ one has $J(u) < \varepsilon$.*

## II. Approximate functional Radon Nikodým theorem

**Theorem.** *Assume $J \ll I$. Then, there exists a sequence $(v_n)_n$ in $\mathcal{B}$ such that for all $f$ in $\mathcal{B}$ one has*

$$J(f) = \lim_n I(f v_n).$$

### Comments and supplementary results

Let us introduce some new notations:

$$AC(I) := \{T : \mathcal{B} \to I\!R | T \text{ linear and positive}, \ T \ll I\}$$

$$R(I) := \{T : \mathcal{B} \to I\!R | T \text{ linear and positive}, \ T \text{ is } I\text{-representable}\}.$$

($T$ is $I$-representable means: there exists a sequence $(v_n)_n$ in $\mathcal{B}$ such that for all $f \in \mathcal{B}$ one has $T(f) = \lim_n I(fv_n)$).

$$SR(I) := \{T : \mathcal{B} \to IR | T \text{ linear and positive, } T \text{ strongly } I\text{-representable}\}.$$

($T$ strongly $I$-representable means : there exists $v$ in $\mathcal{B}$ such that for all $f \in \mathcal{B}$ one has $T(f) = I(fv)$).

Of course $SR(I) \subset R(I)$, and the inclusion is generally strict.

The theorem from above says that $AC(I) \subset R(I)$.

An "exact Radon-Nikodým theorem" would say that $AC(I) = SR(I)$, but this is not true in general. Practically, our result is the best possible. Namely, the inclusion $AC(I) \subset SR(I)$ is generally false (see the example which follows).

**Example.** Take $X = [0,1]$, $\mathcal{B} = \{f : [0,1] \to IR | f \text{ is continuous}\}$. The functional $I : \mathcal{B} \to IR$ is defined as follows: write $Q \cap [0,1] = \{x_n | n \in IN, x_0 = 0, m \neq n \Rightarrow x_m \neq x_n\}$.

Let $(a_n)_n$ be a sequence with $a_n > 0, a_0 = 1$ and $\sum_n a_n$ convergent. Define $I : B \to IR$ via

$$I(f) := \sum_{n=0}^{\infty} a_n f(x_n) = f(0) + \sum_{n=0}^{\infty} a_n f(x_n).$$

The functional $J : \mathcal{B} \to IR$ is defined via

$$J(f) := f(0).$$

Then $J \in AC(I) \backslash SR(I)$.

**Proposition.** *One has $SR(I) \subset AC(I)$ in each of the following situations*:

(i) *The lattice $\mathcal{B}$ consists of bounded functions.*

(ii) *The functional $I$ is a Daniell integral.*

**Open question.** Is it always true that $SR(I) \subset AC(I)$ ?

## III. Exact functional Radon-Nikodým theorem

One must impose supplementary conditions.

We shall work with *Daniell integrals* $I$ and $J$.

Standard procedure gives the spaces:

$$\begin{aligned} L(I) &:= I\text{-integrable functions}; L(J) := J\text{-integrable functions.} \\ L_b(I) &:= \{f \in L(I) \mid f \text{ is bounded}\}; L_b(J) := \{f \in L(J) \mid f \text{ is bounded}\}. \end{aligned}$$

**Proposition.** $J \ll I \Rightarrow L_b(I) \subset L_b(J)$.

The standard extension of $I$ (resp. $J$) to $L(I)$ (resp. $L(J)$) is denoted by $\bar{I}$ (resp. $\bar{J}$).

Let us introduce the following numerical sets (for $u \in L_b(I), u \geq 0$ and $\varepsilon > 0$):

$$\begin{aligned} A\left(\bar{I}, \bar{J}\right)(u) &:= \left\{\frac{\bar{J}(v)}{\bar{I}(v)} | 0 \leq v \leq u,\ v \in L(I),\ \bar{I}(v) > 0\right\} \\ A_\varepsilon\left(\bar{I}, \bar{J}\right)(u) &:= \left\{x \in I\!R \mid |x - a| < \varepsilon \text{ for all } a \in \bar{A}\left(\bar{I}, \bar{J}\right) u)\right\}. \end{aligned}$$

In order to state our theorem (the "exact functional Radon-Nikodým theorem"), we shall make three supplementary assumptions. The first assumption is more "complicated", being sequential and inductive. It consists of a sequence of steps.

**Assumption 1.** The following sequence of conditions (steps) build up this assumption:

**s(1):** There exists a sequence $(h_{n;1})_{n \in I\!N}$ or a finite family $(h_{n;1})_{1 \leq n \leq p_1}$, $0 \leq h_{n;1} \in L(I)$, $\bar{I}(h_{n;1}) > 0$ such that

$$(i_1) \quad \sum_n h_{n;1} = 1.$$

**s(2):** For every $n \in I\!N$ or $1 \leq n \leq p_1$, there exists a sequence $\left(h_{(n,i);2}\right)_{i \in I\!N}$ or a finite family $\left(h_{(n,i);2}\right)_{1 \leq i \leq p_2}$, $0 \leq h_{(n,i);2} \in L(I)$, $\bar{I}\left(h_{(n,i);2}\right) > 0$ such that

$$(\mathrm{i}_2) \quad \sum_i h_{(n,i);2} = h_{n;1}, \text{ all possible } n.$$

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Assuming the step $s(n-1)$ for $n \geq 2$ has been written (this for the family $(h_{\alpha;n-1})_\alpha$, where $\alpha \in I\!N^{n-1}$), we shall write (conventionally) $(\alpha, i_n) \in I\!N^n$ for every $\alpha = (i_1, i_2, \cdots, i_{n-1}) \in I\!N^{n-1}$ and $i_n \in I\!N$. Now we can write the next step :

s(n): For every $\alpha \in I\!N^{n-1}$ in the set of all possible $\alpha$ given by the previous steps, there exists a sequence $\left(h_{(\alpha,i);n}\right)_{i \in I\!N}$ or a finite set $\left(h_{(\alpha,i);n}\right)_{1 \leq i \leq p_n}$ of positive functions in $L(I)$ with $\bar{I}(h_{\beta;n})$ for all possible $\beta$ such that

$$(\mathrm{i}_n) \quad \sum_i h_{(\alpha,i);n} = h_{\alpha;n-1} \text{ all possible } \alpha$$

(this implies $\sum_\beta h_{\beta;n} = 1$). So, $h_{\beta;n} \in L_b(I) \cdots$.

**Assumption 2.** For every natural number $n$ and for every $\alpha \in I\!N^n$ in the set of all possible $\alpha$, one has

$$A_{2^{-n}}\left(\bar{I}, \bar{J}\right)(h_{\alpha;n}) \neq \phi.$$

**Assumption 3.** There exists a number $M > 0$ such that for all $n$ in $I\!N$ and for all $\alpha$ in the set of all possible $\alpha \in I\!N^n$, one has

$$A_{2^{-n}}\left(\bar{I}, \bar{J}\right)(h_{\alpha;n}) \in [-M, M].$$

**Theorem.** *Assume that $I, J$ are Daniell integrals such that $J \ll I$ and the Assumptions* 1,2,3 *are fulfilled. Then there exists a positive bounded function $f$ in $L(I)$ such that*

$$\bar{J}(u) = \bar{I}(fu)$$

*for all $u$ in $L(I)$.*

*The function* $f$ *(called the Radon-Nikodým derivative of* $\bar{J}$ *with respect to* $\bar{I}$*) is* $J$*-almost unique, which means that if* $g \in L(I)$ *is such that* $\bar{J}(u) = \bar{I}(gu)$ *for all* $u$ *in* $L(I)$*. then* $\bar{I}(|f-g|) = 0$.

**Final comments**

A. The theorem in II ("approximate" Radon-Nikodým theorem) appeared in "Rendiconti del Circolo Matematico di Palermo", serie II. vol.48(1999). p.443-450.

The theorem in III ("exact" Radon-Nikodým theorem) will appear in "Studia Mathematica".

B. These results were obtained jointly with the Spanish mathematicians.

Prof.dr. Enrique de Amo Artero (University of Almerìa).

Prof.dr. Manuel Dìaz Carrillo (University of Granada).

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# ON THE EXTENSION OF SOME POSITIVE FUNCTIONALS AND ON EXTENSIBLE REGULAR OPERATORS

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In this paper we give some theorems concerning the extension of continuous positive linear functionals and the extension of positive normal operators. Some spaces of extensible regular operators and of monogenic regular operators are also considered.

## § 1. Extensible continuous positive linear functionals

The first theorem is concerned with positive linear functionals which are continuous with respect to the topology given on an ordered vector space.

**Theorem 1.1.** Let $Z$ be a directed vector space endowed with a locally convex-solid topology and $X$ be a topological vector subspace of $Z$. If $f : X \to IR$ is a continuous positive linear functional, then $f$ extends to a continuous positive linear functional $g : Z \to IR$.

*Proof.* Let $p$ be a continuous solid seminorm on $Z$ such that

$$|f(x)| \leq p(x), \ (\forall x \in X).$$

Putting

$$Q = \{z - a | p(z) < 1; 0 \leq a \in Z\}$$

the set $Q$ is balanced and convex. If $q$ is the Minkowski functional associated to $Q$, then $q$ is a sublinear functional and

$$f(x) \leq q(x), \ (\forall x \in X).$$

By the Hahn-Banach theorem, there exists a linear functional $g$ on $Z$ such that $g|X = f$ and

$$g(z) \leq q(z), \ (\forall z \in Z).$$

From this inequality it results that

$$g(z) < 1,\ \forall z \in Q \tag{1}$$

therefore $g(-nz) < 1, \forall z \in Z, \forall n \in IN$, since $-X_+ \subset Q$. Consequently $g \geq 0$.

On the other hand, $Q$ is an open set and by (1) it results that the linear functional $g$ is continuous.

The theorem is proved.

Let now $Z$ be a directed vector space and $X$ a majorizing vector subspace endowed with a locally convex-solid topology $\tau$. Let $\mathcal{P}$ be the set of all $(\tau)$-continuous solid seminorms on $X$ and consider (as in [2]) the topology $\tilde{\tau}$ on $Z$ defined by the set

$$\tilde{\mathcal{P}} = \{\tilde{p} | p \in \mathcal{P}\}$$

of the seminorms given by the formula

$$\tilde{p}(z) = \inf\{p(x) \mid \pm z \leq x \in X\}, (z \in Z). \tag{2}$$

The topology $\tilde{\tau}$ will be called *the natural extension* of $\tau$ on $Z$.

*Remarks* (i). Every seminorm $\tilde{p}$ given by the formula (2) is a solid seminorm and $\tilde{\tau}|X = \tau$.

(ii) If $\tau'$ is a locally convex-solid topology on $Z$ such that $\tau'|X = \tau$ then $\tau' \leq \tilde{\tau}$.

Indeed, if $p'$ is a $(\tau')$-continuous solid seminorm on $Z$, then there exists a $(\tau)$-continuous solid seminorm $p$ on $X$ such that $p'(x) \leq p(x)$, $\forall x \in X$. If $z \in Z$ then there exists $x \in X$ such that $\pm z \leq x$. We have $p'(z) \leq p'(x) \leq p(x)$ therefore $p'(z) \leq \tilde{p}(z)$, $\forall z \in Z$.

**Theorem 1.2.** Let $Z$ be a directed vector space and $X$ be a majorizing vector subspace endowed with a locally convex-solid topology $\tau$. Let $\tilde{\tau}$ be the natural extension of $\tau$ on $Z$. If $f : X \to IR$ is a $(\tau)$-continuous positive linear functional, then any positive linear functional $g : Z \to IR$ which extends $f$, is $(\tilde{\tau})$-continuous.

*Proof.* Let $p$ be a $(\tau)$-continuous solid seminorm on $X$ such that

$$|f(x)| \leq p(x), \ (\forall x \in X).$$

Let $g : Z \to IR$ be a positive linear functional such that $g|X = f$. Let $v$ be an arbitrary positive element of $Z$. For every number $\varepsilon > 0$ there exists $x \in X$ such that $v \leq x$ and $p(x) < \widetilde{p}(v) + \varepsilon$. It results $g(v) \leq \widetilde{p}(v)$. For an arbitrary element $z$ of $Z$, let $v \in Z$ such that $\pm z \leq v$. We have

$$\pm g(z) \leq g(v) \leq \widetilde{p}(v)$$

therefore $|g(z)| \leq \widetilde{p}(v)$. It results

$$|g(z)| \leq \widetilde{p}(z), \ (\forall z \in Z)$$

and the theorem is proved.

## § 2. Extensible positive normal operators

If $X$ is a vector lattice and $Y$ an $(o)$-complete ordered vector space, we denote by $\mathcal{R}(X, Y)$ the complete vector lattice of all regular operators from $X$ into $Y$.

We shall use the definition of normal operators given in [12] (and in more general conditions, given in [7]).

**Theorem 2.1.** Let $X$ be a vector lattice, $G$ a majorizing vector subspace of $X$ and $Y$ an $(o)$-complete ordered vector space. Let $U_0 : G \to Y$ be a positive normal operator. If $G$ is order dense and $U : X \to Y$ is a positive linear operator such that $U|G = U_0$, then $U$ is a normal operator.

*Proof.* Let $0 \leq V \in \mathcal{R}(X,Y)$ and let $E$ be a totally normal subspace of $X$ such that $V(E) = \{0\}$. Putting $F = E \cap G$, the set $F$ is a totally normal subspace of $G$ and putting $V_0 = V|G$ then $V_0(F) = \{0\}$. From $U_0 \wedge V_0 = 0$ it results $U \wedge V = 0$ (since $G$ is a majorizing subspace of $X$). If $V$ is any abnormal operator then we take into account that $|V|$ is also an abnormal operator.

*Corollary.* If $X$ is a vector lattice, $G$ an order dense majorizing vector subspace of $X$ and $Y$ an $(o)$-complete ordered vector space, then every positive normal operator $U_0 : G \to Y$ extends to a positive normal operator $U : X \to Y$.

*Remark.* If in the theorem 2.1 we consider an Archimedean vector lattice $G$, the Dedekind extension $X$ of $G$ and a complete vector lattice $Y$, then we obtain a theorem of Vekśler [11].

## § 3. Some spaces of regular operators

We shall use the definition of *ideal* introduced in a previous paper [5] in the framework of directed vector spaces (see also [7]).

By *space of type* $(R)$ we mean (as in [8]) a directed vector space which has the Riesz decomposition property.

If $X$ is a space of type $(R)$, $Y$ an $(o)$-complete ordered vector space, $G$ a vector subspace of $X$ and $\mathcal{Z}$ a vector subspace of $\mathcal{R}(X,Y)$ we shall denote

$$\mathcal{Z}_e(G,Y) = \{U \in \mathcal{R}(G,Y) \mid \exists\, V \in \mathcal{Z}, V|G = U\}.$$

The following theorem generalizes a theorem given in [7] and the proof is similar.

**Theorem 3.1.** Let $X$ be a spaces of type $(R)$ and $Y$ an $(o)$-complete ordered vector space. If $G$ is an ideal of $X$ and $\mathcal{Z}$ an ideal of $\mathcal{R}(X,Y)$ then $\mathcal{Z}_e(G,Y)$ is an ideal of $\mathcal{R}(G,Y)$.

*Examples* (i). If $X$ is a space of type $(R)$ and $Y$ an $(o)$-complete ordered vector space, then the set $\mathcal{Z}$ of all $(\omega)$-continuous regular operators from $X$ into $Y$, is an ideal (even a band) of the space $\mathcal{R}(X,Y)$, (see [8]);

(ii). If $X$ is a Banach lattice and $Y$ a space of type $(KB)$, then the set $\mathcal{Z}$ of all summable operators (from $X$ into $Y$) is an ideal of $\mathcal{R}(X,Y)$, (see [3]).

We shall now use the following definition given in [10] but considered in the sequel in more general conditions.

Let $X$ be a space of type $(R)$ and $Y$ a complete vector lattice. Let $G$ be a vector subspace of $X$ and $\mathcal{Z}$ be a vector sublattice of $\mathcal{R}(X,Y)$. A positive operator $U \in \mathcal{Z}$ is said to be *monogenic* (with respect to $\mathcal{Z}$ and $G$) if from $0 \leq V \in \mathcal{Z}$ and $V|G = U|G$ it results $V = U$.

We shall denote by $\mathcal{M}(\mathcal{Z}, G)$ the set of all operators $U \in \mathcal{Z}$ such that $|U|$ be monogenic (with respect to $\mathcal{Z}$ and $G$).

*Remark.* If $G$ is a full vector subspace of $X$, if $U \in \mathcal{M}(\mathcal{Z}, G)$ and $U|G \geq 0$ then $U \geq 0$.

Indeed if $U \in \mathcal{M}(\mathcal{Z}, G)$ then $U_+ \in \mathcal{Z}$ and if $0 \leq a \in G$ then $U_+(a) = U(a)$.

In the following lemma, by ***directed vector subspace*** of a directed vector space $X$, we mean (as in [9]) a vector subspace $G$ with the property: if $a \in G$ and $a \leq x \in X_+$ then there exists $b \in G_+$ such that $a \leq b \leq x$.

*Lemma.* Let $X$ be a space of type $(R)$, $G$ a majorizing directed vector subspace of $X$ and $Y$ an $(o)$-complete ordered vector space. If $\mathcal{Z}$ is a full vector sublattice of $\mathcal{R}(X,Y)$ and $O \leq U \in \mathcal{Z}$, then the following two conditions are equivalent

(i). $U \in \mathcal{M}(\mathcal{Z}, G)$;

(ii). $0 \leq V \in \mathcal{Z}, U|G \leq V|G \Rightarrow U \leq V$.

*Proof.* (i) $\Rightarrow$(ii). Let $U \in \mathcal{M}(\mathcal{Z}, G)$ and $0 \leq V \in \mathcal{Z}$ such that $U|G \leq V|G$. Putting

$$P(x) = \inf\{V(z) \,|0,\ x \leq z \in X\},\ (x \in X)$$

we obtain a sublinear operator $P : X \to Y$.

We also have

(3) $$P(a) = \inf\{V(b) \,|0, a \leq b \in G\},\ (\forall a \in G).$$

Indeed, denoting by $A$ the set in the right side of the formula (3), we have obviously

$P(a) \leq \inf A$. On the other hand, if $a \in G$ and $0, a \leq z \in X$, then there exists $b \in G$ such that $0, a \leq b \leq z$ and we have $V(b) \leq V(z)$. Consequently $\inf A \leq P(a)$.

Using the equality (3), it is easily seen that

$$U(a) \leq P(a), \ (\forall a \in G).$$

Therefore there exists a linear operator $W : X \to Y$ such that $W|G = U|G$ and

$$W(x) \leq P(x), \ (\forall x \in X).$$

From this inequality it results $0 \leq W \leq V$, therefore $W \in \mathcal{Z}$. From $0 \leq W \in \mathcal{Z}$ and $W|G = U|G$ it results $W = U$ since $U \in \mathcal{M}(\mathcal{Z}, G)$. Consequently $U \leq V$.

(ii) $\Rightarrow$(i). If $0 \leq V \in \mathcal{Z}$ and $V|G = U|G$, then from $U|G \leq V|G$ and (ii) it results $U \leq V$. On the other hand, from $0 \leq U \leq V$ and $U|G = V|G$ it results $V \leq U$ since if $x \in X_+$ and $x \leq a \in G_+$ then $U(a - x) \leq V(a - x)$ therefore $V(x) \leq U(x)$. Consequently $U = V$ that is $U \in \mathcal{M}(\mathcal{Z}, G)$.

The lemma is proved.

The following theorem generalizes a theorem in [10] (and [4]).

**Theorem 3.2.** Let $X$ be a space of type $(R)$, $G$ a majorizing directed vector subspace of $X$ and $Y$ an $(o)$-complete ordered vector space. If $\mathcal{Z}$ is a band in the space $\mathcal{R}(X, Y)$ then $\mathcal{M}(\mathcal{Z}, G)$ is a band in the space $\mathcal{Z}$.

*Proof.* Using the previous lemma, we establish (as in [4]) that if $0 \leq U_1, U_2 \in \mathcal{M}(\mathcal{Z}, G)$ and $0 < \alpha, \beta \in I\!R$ then $\alpha U_1 + \beta U_2 \in \mathcal{M}(\mathcal{Z}, G)$, if $0 \leq U_1 \leq U_2 \in \mathcal{M}(\mathcal{Z}, G)$ and $U_1 \in \mathcal{Z}$ then $U_1 \in \mathcal{M}(\mathcal{Z}, G)$ and if $U_\delta \underset{\delta \in \Delta}{\uparrow} U$ in $\mathcal{Z}$ and $0 \leq U_\delta \in \mathcal{M}(\mathcal{Z}, G)$ then $U \in \mathcal{M}(\mathcal{Z}, G)$. After that, we take into account the definition of $\mathcal{M}(\mathcal{Z}, G)$.

*Remark.* If the conditions of the above theorem are satisfied, then the set $\mathcal{M}(\mathcal{Z}, G)$ is a component [1] of the space $\mathcal{R}(X, Y)$.

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# Some Remarks on Lattice-Subspaces

Nicolae Dăneţ

## 1 Preliminaries on lattice-subspaces

A vector subspace $X$ of an ordered vector space $E$ is said to be a *lattice-subspace* if $X$ under the induced ordering is a vector lattice. For each $x, y \in X$ we shall denote by $x \bigtriangledown y$ the supremum and by $x \bigtriangleup y$ the infimum of $\{x, y\}$ in $X$. If $x \vee y$ and $x \wedge y$, the supremum and the infimum of $\{x, y\}$ in $E$, exist, then we have $x \vee y \leq x \bigtriangledown y$ and $x \bigtriangleup y \leq x \wedge y$. If $E$ is a vector lattice and $x \vee y = x \bigtriangledown y$ for all $x, y \in X$, then $X$ is a vector sublattice of $E$. Obviously, every vector sublattice is a lattice-subspace, but, in general, the converse is not true. The difficulty for study of lattice-subspaces is that $x \bigtriangledown y$ in $X$ depends on the subspace $X$. In lattice-subspace we have the induce ordering but the lattice structure is not the induced one.

Let $E$ an ordered Banach space. A sequence $\{e_n\}$ is called a *positive basis* of $E$ if $\{e_n\}$ is a (Schauder) basis of $E$ and the positive cone $E_+$ is equal with $\{x \in E \mid x = \sum_{n=1}^{\infty} \lambda_n e_n, \lambda_n \geq 0 \text{ for all } n\}$. A positive basis in $E$ is unique in the sense that if $\{b_n\}$ is another positive basis of $E$, then each element of $\{b_n\}$ is a positive multiple of an element of $\{e_n\}$. If $\{e_n\}$ is a positive unconditional basis of $E$, then the lattice operations in $E$ are given by

$$x \vee y = \sum_{n=1}^{\infty} (\lambda_n \vee \mu_n) e_n \quad \text{and} \quad x \wedge y = \sum_{n=1}^{\infty} (\lambda_n \wedge \mu_n) e_n$$

for each $x = \sum_{n=1}^{\infty} \lambda_n e_n$, $y = \sum_{n=1}^{\infty} \mu_n e_n$ in $E$.

For the study of finite-dimensional lattice-subspaces the following theorem is very important (see [1] and [7]).

**Theorem 1** *A finite-dimensional ordered vector space $E$ is a vector lattice if and only if $E$ has a positive basis.*

Lattice-subspaces have applications in economics (see [2], [3] and [9]).

## 2 The lattice-subspace $X(x) = [x^+, x^-]$

Let $E$ be a *vector lattice*. Every element $x \in E$ has the decomposition $x = x^+ - x^-$. If $x$ has another decomposition $x = u - v$, with $u, v \geq 0$, then $x^+ \leq u$ and $x^- \leq v$. This means that the decomposition $x = x^+ - x^-$ is a minimal decomposition among all the positive decompositions of $x$. In the particular case when $u = x^+$ and $v = x^-$, in plus, we have $u \wedge v = x^+ \wedge x^- = 0$. Conversely, if $x = u - v$, with $u \wedge v = 0$, then $u = x^+$ and $v = x^-$([5], p.73).

Let $x$ be an element in $E$ such that $x = x^+ - x^-$, with $x^+ \neq 0$ and $x^- \neq 0$. Then $x^+$ and $x^-$ are *linearly independent* vectors in $E$. (Indeed, if $\lambda_1, \lambda_2 \in \mathbb{R}$ are such that $\lambda_1 x^+ + \lambda_2 x^- = 0$, then, using the fact that $\lambda_1 x^+ \perp \lambda_2 x^-$, we have

$$0 = |\lambda_1 x^+ + \lambda_2 x^-| = |\lambda_1| x^+ + |\lambda_2| x^- \geq |\lambda_1| x^+,\ \ |\lambda_2| x^- \geq 0.$$

Therefore $|\lambda_1| x^+ = 0$, $|\lambda_2| x^- = 0$, from where it results that $\lambda_1 = \lambda_2 = 0$.)

Denote by $X(x) = [x^+, x^-]$, or simply by $X$, the (closed) linear subspace generated by $x^+$ and $x^-$ endowed with the induced ordering from $E$. Then the set $\{x^+, x^-\}$ is a *positive basis* for $X$. To prove this assertion, let $\lambda_1, \lambda_2 \in \mathbb{R}$ be such that $\lambda_1 x^+ + \lambda_2 x^- \geq 0$. We must show that $\lambda_1, \lambda_2 \geq 0$. Using again the orthogonality $\lambda_1 x^+ \perp \lambda_2 x^-$, we obtain

$$\lambda_1 x^+ + \lambda_2 x^- = |\lambda_1 x^+ + \lambda_2 x^-| = |\lambda_1| x^+ + |\lambda_2| x^-.$$

The uniqueness of the representation of a vector in the basis $\{x^+, x^-\}$ implies the equalities $\lambda_1 = |\lambda_1|$, $\lambda_2 = |\lambda_2|$, from where it results $\lambda_1 \geq 0$, $\lambda_2 \geq 0$.

Since $X = [x^+, x^-]$ has the positive basis $\{x^+, x^-\}$, $X$ is a lattice-subspace in $E$, and the lattice operations are given by the following formulas: if $z = \lambda_1 x^+ + \lambda_2 x^-$ and $w = \mu_1 x^+ + \mu_2 x^-$, then

$$\begin{aligned} z \bigtriangledown w &= \max\{\lambda_1, \mu_1\} x^+ + \max\{\lambda_2, \mu_2\} x^- \\ z \bigtriangleup w &= \min\{\lambda_1, \mu_1\} x^+ + \min\{\lambda_2, \mu_2\} x^- \\ |z|_X &= |\lambda_1| x^+ + |\lambda_2| x^- \end{aligned}$$

In fact, $X = [x^+, x^-]$ is a *sublattice* of $E$. This results from the observation that, for every $z = \lambda_1 x^+ + \lambda_2 x^- \in X$, we have

$$|z| = |\lambda_1 x^+ + \lambda_2 x^-| = |\lambda_1| x^+ + |\lambda_2| x^- = |z|_X.$$

Let $E$ be a *directed vector space* and $x$ an element in $E$. Let $x = x_1 - x_2$ be a positive decomposition of $x$ such that $x_1, x_2$ are linearly independent. We can consider $X(x) = [x_1, x_2]$ and construct a positive basis $\{b_1, b_2\}$

in $X(x)$. (In 2-dimensional case such a basis always exists, see [7].) If $z = \lambda_1 b_1 + \lambda_2 b_2$ is an element in $X(x)$, then we can define $|z|_X = |\lambda_1| b_1 + |\lambda_2| b_2$. This construction can be useful in some situations, but, in general, we can not use $|z|_X$ like a substitute for the modulus of $z$ because the construction depends of the initial positive decomposition of $x$.

# 3 An algorithm to determine the lattice-subspaces in $\mathbb{R}^m$

For applications in economics it is important to determine whether or not a given set of $n$ linearly independent positive vectors of $\mathbb{R}^m$ $(n < m)$ generates a lattice-subspace. Such an algorithm is describe in [1].

In this section we obtain a simple form of this algorithm which permit to observe that we can use the classical Gauss-Jordan algorithm to decide if a given collection of positive vectors generates a lattice-subspace.

Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ be $n$ linearly independent positive vectors of $\mathbb{R}^m$, where $1 \leq n < m$, and denote by $X = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n]$ the $n$-dimensional vector subspace they generate. We shall write these vectors in column form

$$\mathbf{x}_j = \begin{bmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{mj} \end{bmatrix}, \quad j = 1, \dots, n.$$

Let $A$ be the matrix of type $m \times n$ whose columns are the vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$. Hence

$$A = \begin{bmatrix} x_{11} & x_{12} & & x_{1n} \\ x_{21} & x_{22} & & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{m1} & x_{m2} & & x_{mn} \end{bmatrix} \in \mathcal{M}_{m,n}(\mathbb{R}).$$

Since $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ are linearly independent vectors, the rank of the matrix $A$ is equal to $n$. Consider now the transpose matrix of $A$,

$$A^T = \begin{bmatrix} x_{11} & x_{21} & \dots & x_{m1} \\ x_{12} & x_{22} & \dots & x_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1n} & x_{2n} & \dots & x_{mn} \end{bmatrix} \in \mathcal{M}_{n,m}(\mathbb{R}).$$

The columns of the matrix $A^T$ are vectors of $\mathbb{R}^n$ which will be denoted by $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_m$. Therefore

$$\mathbf{y}_i = \begin{bmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{in} \end{bmatrix}, \quad i = 1, \ldots, m.$$

Since the matrix $A^T$ has the rank $n$, and $1 \leq n < m$, it follows that there exist $n$ linearly independent vectors among the vectors $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_m$ which are denoted by $\mathbf{y}_{i_1}, \mathbf{y}_{i_2}, \ldots, \mathbf{y}_{i_n}$. Then the rest of the vectors $\mathbf{y}_i$, where $i \in \{1, 2, \ldots .m\} \setminus \{i_1, i_2. \ldots, i_n\}$, can be written like a linear combination of the vectors $\mathbf{y}_{i_1}, \mathbf{y}_{i_2}, \ldots, \mathbf{y}_{i_n}$, i.e.,

$$\mathbf{y}_i = \sum_{k=1}^{n} \xi_{ik} \mathbf{y}_{i_k}.$$

If the vectors $\mathbf{y}_i$, $i \in \{1, 2, \ldots .m\} \setminus \{i_1, i_2. \ldots, i_n\}$, belong to the cone generated by the vectors $\mathbf{y}_{i_1}, \mathbf{y}_{i_2}, \ldots, \mathbf{y}_{i_n}$, which means that all the coefficients $\xi_{ik}$ are nonnegative, then the set $\{i_1, i_2. \ldots, i_n\}$ is called a *fundamental set of indices* for the collection of vectors.$\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^m_+$.

In [1] it is proved the following theorem.

**Theorem 2** *The vector subspace $X$ is a lattice-subspace of $\mathbb{R}^m$ if and only if the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ admit a fundamental set of indices $\{i_1, i_2. \ldots, i_n\}$.*

In the sequel we shall show how we can construct, in a easy manner for using a computer, a positive basis for $X$, if $\{i_1, i_2. \ldots, i_n\}$ is a fundamental set of indices for the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$.

Let $B$ be the submatrix of the transpose matrix $A^T$ which has the vectors $\mathbf{y}_{i_1}, \mathbf{y}_{i_2}, \ldots, \mathbf{y}_{i_n}$ as columns. Since these vectors are linearly independent, the matrix $B$ has a nonzero determinant and therefore it is invertible. Define now the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ of $\mathbb{R}^m$, symbolically, by the formula

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_n \end{bmatrix} = B^{-1} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}.$$

More precisely, if we denote $C^T = B^{-1}$, and $C = [c_{kj}] \in \mathcal{M}_{n,n}(\mathbb{R})$, then we have the following vectorial relations

$$\mathbf{e}_j = \sum_{k=1}^{n} c_{kj} \mathbf{x}_k, \quad j = 1, \ldots, n, \tag{1}$$

or, on components,

$$e_{ij} = \sum_{k=1}^{n} c_{kj} x_{ik}, \quad j = 1, \dots, n, \quad i = 1, \dots, m. \tag{2}$$

We shall prove that the vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ form a *positive basis* for $X$. Firstly we observe that the relations (1) show that the vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ belong to $X$ and, since $\det C \neq 0$, they form a *basis* for this subspace.

In the sequel we shall prove that $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is a *positive* basis of $X$. For simplicity of notation we assume that the vectors $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ are linearly independent, and $\mathbf{y}_{n+1}, \dots, \mathbf{y}_m$ have the following expansions

$$\begin{cases} \mathbf{y}_{n+1} = \xi_{n+1,1}\mathbf{y}_1 + \xi_{n+1,2}\mathbf{y}_2 + \cdots + \xi_{n+1,n}\mathbf{y}_n \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \mathbf{y}_m = \xi_{m,1}\mathbf{y}_1 + \xi_{m,2}\mathbf{y}_2 + \cdots + \xi_{m,n}\mathbf{y}_n \end{cases}$$

or

$$\mathbf{y}_p = \sum_{s=1}^{n} \xi_{ps} \mathbf{y}_s, \quad p = n+1, \dots, m, \tag{3}$$

where $\xi_{pk} \geq 0$. Using these notations we shall prove that the vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ have the following form

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \xi_{n+1,1} \\ \vdots \\ \xi_{m,1} \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ \xi_{n+1,2} \\ \vdots \\ \xi_{m,2} \end{bmatrix}, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \xi_{n+1,n} \\ \vdots \\ \xi_{m,n} \end{bmatrix}. \tag{4}$$

With other words, using the relations (2), we have to prove that

$$e_{ij} = \sum_{k=1}^{n} c_{kj} x_{ik} = \delta_{ij}, \quad j = 1, \dots, n, \quad i = 1, \dots, n. \tag{5}$$

$$e_{pj} = \sum_{k=1}^{n} c_{kj} x_{pk} = \xi_{pj} \quad j = 1, \dots, n, \quad p = n+1, \dots, m. \tag{6}$$

For proving these relations observe firstly that $C^T = B^{-1}$ implies $C^T B = I_n$, and then, between the components, we have

$$\sum_{k=1}^{n} c^T_{jk} b_{ki} = \delta_{ij}, \quad i, j = 1, \dots, n.$$

Since $b_{ki} = x_{ik}$, we obtain

$$\sum_{k=1}^{n} c_{kj} x_{ik} = \delta_{ij}, \quad i, j = 1, \dots, n.$$

The relations (5) are thus proved.

The vectorial relations (3), written on components, give

$$x_{pk} = \sum_{s=1}^{n} \xi_{ps} x_{sk}, \quad p = n+1, \dots, m, \quad k = 1, \dots, n.$$

Then we have

$$\begin{aligned} e_{pj} &= \sum_{k=1}^{n} c_{kj} x_{pk} = \sum_{k=1}^{n} c_{kj} \left( \sum_{s=1}^{n} \xi_{ps} x_{sk} \right) = \sum_{s=1}^{n} \xi_{ps} \left( \sum_{k=1}^{n} c_{kj} x_{sk} \right) \\ &= \sum_{s=1}^{n} \xi_{ps} \delta_{js} = \xi_{pj} \geq 0. \end{aligned}$$

and the relations (4) are completely proved. They show that $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ are positive vectors and form a positive basis for $X$.

To decide if we have a fundamental set of indices for the vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n \in \mathbb{R}^m_+$ we can transform the matrix $A^T$ to reduce row echelon form using the well-known Gauss-Jordan algorithm ([4], p.364). With the aid of this algorithm it is easy to decide when a collection of positive operators $T_1, T_2, \dots, T_p \in L(\mathbb{R}^n, \mathbb{R}^m)$ determine a lattice-subspace.

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# A Hahn-Banach Type Theorem for Riesz Homomorphisms

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**Abstract**

In this paper, we give an extension theorem for Riesz homomorphisms, along the line of Hahn-Banach result. As a consequence, we can obtain the classic theorem of W. A. J. Luxemburg and A. R. Schep (1979). This theorem, actually a Kantorovich type theorem, was reproved in 1985 by Z. Lipecki. The idea of our result is due to Lipecki's proof.

## 1 Preliminaries

When we say a *Riesz homomorpshism* between two vector lattices $X$ and $Y$, we understand a linear operator $T : X \to Y$ such that, for all $x_1, x_2$ in $X$,

$$T(x_1 \vee x_2) = T(x_1) \vee T(x_2)(\text{or equivalently,}T(x_1 \wedge x_2) = T(x_1) \wedge T(x_2)).$$

Hence, the Riesz homomorphisms are the morphisms in the vector lattices setting. (For a monographic study of these operators and an interesting view on the problems of this theory, see [D4].)

We also recall that a *majorizing subspace* of an ordered vector space $X$ is a vector subspace $G$ of $X$, such that for any $x$ in $X$ there exists $v$ in $G$ satisfying $x \leq v$.

If $G$ is a majorizing subspace of $X$, $Y$ is a complete vector lattice and $T : G \to Y$ is a positive linear operator, then we can consider the function $\bar{T} : X \to Y$, associated to $T$ by the following formula:

$$\bar{T}(x) = \sup\{T(v) \mid v \in G, v \leq x\}, \text{ for each } x \text{ in } X.$$

This function, introduced in a particular case, in 1923 by F. Riesz, has the following properties:

1) $\bar{T}$ is sublinear; 2) $\bar{T}$ is increasing and positive; 3) $\bar{T} = T$ on G; 4) $S(x) \leq \bar{T}(x)$, for all $x$ in $X$ and for any $S : X \to Y$ a positive linear extension of $T$.

## 2 The Main Results

The problem of the extension of the Riesz homomorphisms was solved for the first time in 1962, by A. Hayes, in the lattice groups setting. The Riesz homomorphisms are positive operators, but the classic proofs for the extension of positive linear operators didn't assure that the extension of a Riesz homomorphism is also a Riesz homomorphism.

In 1979, W. A. J. Luxemburg and A. R. Schep proved that this is true if the domain of the operator which must be extended is a majorizing sublattice $G$ of a given vector lattice $X$ and the range space $Y$ is a complete vector lattice. Later, in 1985, Z. Lipecki gave another proof for Luxemburg and Schep's result.

In [D3] we investigated some conditions in which more general results of extension should be valid. The obtained results completed some theorems of [L1], [C1], [D1] and [D2].

For example, the following result is a first step in the extension of Riesz homomorphisms.

**Proposition 1** ([D3] and [D4]) *Let $X$ and $Y$ be two vector lattices, $G$ a vector sublattice of $X$, $M$ a wedge of $X$ closed under finite suprema and containing $G$, $H = Sp(M)$ and $P : M \to Y$ a function, such that it is additive and positively homogenous on $M$. Let also $T : G \to Y$ be a Riesz homomorphism such that $T = P$ on $G$. Then:*

*a) $H$ is a vector sublattice of $X$;*

*b) there exists a Riesz homomorphism $S : H \to Y$ which extends $T$ and such that $S = P$ on $M$ iff $P(z_1 \vee z_2) = P(z_1) \vee P(z_2)$, for all $z_1, z_2$ in $M$.*

Note that the operator $S$ is defined by $S(z_1 - z_2) = P(z_1) - P(z_2)$, for each $z_1, z_2 \in M$.

The following theorem is *the main result* of this paper:

**Theorem 2** *Let $X$ and $Y$ be vector lattices, $G$ a vector sublattice of $X$, $T : G \to Y$ a Riesz homomorphism and $P : X \to Y$ a positively homogenous operator such that $P = T$ on $G$, $P(x_1 \vee x_2) = P(x_1) \vee P(x_2)$, for all $x_1, x_2$ in $X$ and $P(v + x) = T(v) + P(x)$, for all $v$ in $G$ and $x$ in $X$. Then there exists a Riesz homomorphism $S : X \to Y$, which extends $T$.*

**Proof.** First we will extend $T$ to the vector sublattice generated by $G$ and an element $x_0$ in $X \setminus G$.

Let $M = \left\{ \bigvee_{i=1}^{n} (v_i + \alpha_i x_0) | v_i \in G, \alpha_i \in \mathbb{R}_+, i = \overline{1,n}, n \in \mathbb{N} \right\}$.

Obviously, $M$ is a wedge in $X$, closed under finite suprema and containing $G$. Moreover, $P$ is additive on $M$.

If $H$ is the vector sublattice generated by $G$ and $x_0$, then $H = M - M$.

With Proposition 1, there exists a Riesz homomorphism $T_1 : H \to Y$ which extends $T$, such that $T_1(x_0) = P(x_0)$. We recall that $T_1(z_1 - z_2) = P(z_1) - P(z_2)$, for all $z_1, z_2$ in $M$. (Also, we remark that $T_1$ is uniquely determined.)

Now, by a standard application of Zorn's lemma, we obtain a Riesz homomorphism which extends $T$ to the whole $X$. ■

Applying Theorem 2, for $Y$ a complete vector lattice, $G$ a majorizing sublattice of $X$ and $P = \bar{T}$, we obtain the following result of W. A. J. Luxemburg and A. R. Schep concerning the extension of the Riesz homomorphisms (see also the proof of this result, due to Z. Lipecki).

**Theorem 3** ([L3] and [L2]) *If $X$ and $Y$ are two vector lattices, $G$ is a majorizing sublattice of $X$ and $T : G \to Y$ is a Riesz homomorphism, then there exists a Riesz homomorphism $S : X \to Y$, which extends $T$.*

Unlike Luxemburg and Schep's proof for Theorem 3, which is very long and technical, our proof is very simple.

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# OPERATEURS (*o*) – CONVEXES, DÉRIVABLES

**M. Gavrilă**

On connait bien qu'une fonction réelle de variable réelle, $f$, qui est dérivable est une fonction convexe si et seulement si sa dérivée $f'$ est croissante.

Cette propriété ne reste plus valable si la fonction $f$ a comme domaine de definition un espace réticulé Banach $X$, ayant la dimension, $\dim X \geq 2$. Plus exactément il ya des fonctions $f : X \to \Re$ de classe $C^1(X)$ qui sont (*o*) – convexes et leur dérivée n'est pas croissante, mais il y a aussi des exemples de fonctions $f : X \to \Re$ de classe $C^1(X)$ pour lesquelles la dérivée est croissante, sans qu'elles soient (*o*) – convexes.

Dans cette note on donné une caractérisation de (*o*) – convexité en termes de monotonie de la dérivée.

**DEFINITION 1**: Soient $X$, $Y$ deux espaces Banach, $X$ un espace linéaire dirigé, $x_0 \in E$. On dit que un opérateur $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ a une *différentielle Gâteaux* dans $x_0$, si $(\forall)\ s \in X\ (\exists)\ y(x_0, s) \in Y$, donné par

$$y(x_0, s) = \lim_{\lambda \downarrow 0} \frac{f(x_0 + \lambda s) - f(x_0)}{\lambda}$$

On note $f'(x_0) : X \to Y$ l'opérateur linéaire donné par la formule

$$f'(x_0)(s) = y(x_0, s) := \langle f'(x_0), s \rangle$$

$f'(x_0)$ s'appelle la différentielle Gâteaux de $f$ dans $x_0$.

On note $f': X \to \mathcal{L}(X,Y)$

$x_0 \to f'(x_0)$

$f'$ s'appelle l'opérateur gradient de f (l'opérateur de la derivée de f).

Un opérateur $g: E \subset X \to \mathcal{L}(X,Y)$ s'appelle *gradient* si $(\exists)$ $f: E \to Y$ différentielle sur le domaine $E$ telle que $g=f'$.

**DEFINITION 2**: Soient $X$, $Y$ deux espaces Banach qui sont aussi les espaces linéaires ordonnées, $E \subset X$ un sous-ensemble convexe et $f: E \to Y$.

On dit que $f$ est *(o)-convexe* si

$$f((1-\lambda)x+\lambda y) \le (1-\lambda)f(x)+\lambda f(y)$$

$(\forall)$ $x,y \in E$ des élémentes comparables et $(\forall)$ $\lambda \in [0,1]$.

**DEFINITION 3**: Soit $X$ un espace linéaire reticulé et $Y$ un espace linéaire ordonné. L'opérateur $f: X \to Y$ s'appelle *monotone-symetrique* ((s)-monotone croissant) si pour tous éléments $x,y,y' \in E$ qui verifient l'inégalité $|y| \le |y'|$, on a

$$f(x+y)+f(x-y) \le f(x+y')+f(x-y').$$

**DEFINITION 4**: Soit $X$ un espace linéaire reticulé et $Y$ un espace linéaire. Un opérateur $f: X \to Y$ s'appelle *ortogonal-additif* si pour $x,y \in X$ $x \perp y$ $(|x| \wedge |y| = 0)$ on a la relation

$$f(x+y)=f(x)+f(y).$$

Pour $a \in E$ on considère l'opérateur $f_a: X \to Y$ définie par la relation

$$f_a(x)=f(a+x)-f(a),\ (\forall)\ x \in X.$$

Un opérateur $f : X \to Y$ s'appelle *monotone-croissante* si $f(x)>f(y)$ pour tous $x,y \in X$, $x \geq y$.

## THÉORÈME 1:

Soient $X$, $Y$ deux espaces Banach telle que $X$ est aussi un espace linéaire reticulé, $Y$ un espace linéaire ordonné, $E \subset X$ un sous-espace linéaire reticulé, $f : E \to Y$ une différentielle Gâteaux, $g$ le gradient de $f$. Si $f$ est un opérateur (s)-monotone croissant, alors:

1) pour tout $a \in E$, $g_a$ est ortogonal-additif;
2) $g$ est monotone-croissante;
3) si $y \perp z$ alors $<g(y), z>=<g(0), z>$, et plus généralement

$$<g(x+y), z>=<g(x), z>, \ (\forall)\ x \in E.$$

**Démonstration**:

1) Soint $a \in E$ et $f_a(x)=f(a+x)-f(a)$, $(\forall)\ x \in E$.

On montre que $f_a$ est un operator ortogonal – additif et donc

$$f_a(y_1+y_2)=f_a(y_1)+f_a(y_2)$$

pour tout $y_1,y_2 \in E$, $y_1 \perp y_2$.

En utilisant la définition de $f_a$, il résulte que

$$f(a+y_1+y_2)=f(a+y_1)+f(a+y_2)-f(a)$$

Analoque, pour tout $h \in E$ et $t \in (0,\infty)$ nous avons que $f_{a+th}$ est ortogonal–additif, et donc

$$f_{a+th}(y_1+y_2)=f_{a+th}(y_1)+f_{a+th}(y_2),\ \forall y_1,y_2 \in E,\ y_1 \perp y_2.$$

Il resulte

$$f(a+th+y_1+y_2)=f(a+th+y_1)+f(a+th+y_2)-f(a+th)$$

et donc:

$$f(a+th+y_1+y_2)-f(a+y_1+y_2)=[f(a+th+y_1)-f(a+y_1)]+[f(a+th+y_2)-f(a+y_2)]-$$

$$-[f(a+th)-f(a)].$$

Par conséquence

$$\lim_{t\to 0}\frac{f(a+y_1+y_2+th)-f(a+y_1+y_2)}{t}=\lim_{t\to 0}\frac{f(a+y_1+th)-f(a+y_1)}{t}+$$
$$\lim_{t\to 0}\frac{f(a+y_2+th)-f(a+y_2)}{t}+\lim_{t\to 0}\frac{f(a+th)-f(a)}{t}$$

d'ou il resulte

$$g(a+y_1+y_2)=g(a+y_1)+g(a+y_2)-g(a)$$

et donc

$$g_a(y_1+y_2)=g_a(y_1)+g_a(y_2)$$

2) Si $x, y \in E$ telle que $x\leq z$, alors $x^+\leq z^+$ et $x^-\geq z^-$.

Il résulte que

$$x^++ty\leq z^++ty,\ \forall y\in E_+,\ \forall\, t\in(0,\infty),$$

et donc

$$\left|x^++ty-z^+\right|\leq\left|z^++ty-x^+\right|$$

Parce que $f$ est (s) – monotone croissante, il résulte:

$$f(x^++ty)-f(x^+)\leq f(z^++ty)-f(z^+).$$

et donc

$$< g(x^+), y >\leq< g(z^+), y >,\ \forall y\in E_+.$$

De manière analoque $-x^-\leq -z^-$

et donc

$$-x^-+ty\leq -z^-+ty,\ \forall y\in E_+,\ \forall\, t\in(0,\infty).$$

Il résulte que

$$\left|(-x^-)+ty-(-z^-)\right|\leq\left|(-z^-)+ty-(-x^-)\right|$$

et parceque $f$ est ($s$)-monotone croisante, ou obtien:

$$f(-x^-+ty)-f(-x^-)\leq f(-z^-+ty)-f(-z^-)$$

Donc

$$< g(-x^-), y > \le < g(-z^-), y >, \ \forall y \in E_+.$$

Parceque l'operateur $g_0$ est ortogonal – additif il résulte

$$g_0(x)=g_0(x^+)+g(-x^-) \Leftrightarrow g(x)-g(0)=g(x^+)-g(0)+g(-x^-)-g(0)$$

De la même manière on demontré

$$g(z)-g(0)=g(z^+)-g(0)+g(-z^-)-g(0)$$

et donc

$$< g(x), y > \le < g(z), y >, \ \forall y \in E_+.$$

3) Soient $y, z \in E$ telle que $y \perp z$. Puisque $f_0$ est ortogonal-additif et $y \perp tz, \ \forall t \in (0,\infty)$,

il résulte que

$$f_0(y+tz)=f_0(y)+f_0(tz) \Leftrightarrow f(y+tz)-f(y)=f(tz)-f(0) \Leftrightarrow$$

$$\lim_{t\to 0}\frac{f(y+tz)-f(y)}{t}=\lim_{t\to 0}\frac{f(tz)-f(0)}{t} \Leftrightarrow < g(y), z > = < g(0), z >.$$

Analoque en utilisant $f_x$ est ortogonal-additif, il résulte que

$$< g(x+y), z > = < g(x), z > \ \forall x \in E.$$

**THÉORÈME 2**.

Soit $X$ un espace Banach qui est aussi un espace linéaire reticulé, $\mathbf{E} \subset \mathbf{X}$ un sous-espace linéaire reticulé, $\mathbf{g} : \mathbf{E} \to \mathcal{L}(\mathbf{X}, \mathfrak{R})$ un opérateur gradient, continu qui vérifie:

1) pour tout $a \in E$, $g_a$ est ortogonal-additif;
2) $g$ est monotone-croissante;
3) si $y \perp z$ alors $< g(y), z > = < g(0), z >$, et plus généralement

$< g(x+y), z > = < g(x), z >$, $(\forall)\ x \in E$.

Si $\mathbf{f(x)} = \int_0^1 <\mathbf{g(tx), x}> \mathbf{dt}$ alors

1) $a \in E, f_a$ est ortogonal-additif;

2) $f$ est (s)-monotone croisante.

**<u>Démonstration</u>**: Soient $y, z \in E$ , $y \perp z$ et $a \in E$ .

Puisque $f_a(y+z)=f(a+y+z)-f(a)=$

$$\int_0^1 < g(t(a+y+z)), a+y+z > dt - \int_0^1 < g(ta), a > dt = \int_0^1 < g(t(a+y+z)) - g(ta), a > dt$$

$$\int_0^1 < g(t(a+y+z)), y > dt + \int_0^1 < g(t(a+y+z)), z > dt$$

et g est ortogonal-additif, il résulte que

$g_{ta}(ty+tz)=g_{ta}(ty)+g_{ta}(tz), \ \forall t \in [0,1] \Leftrightarrow g(ta+ty+tz)-g(ta) = g(ta+ty)+g(ta+tz)-2g(ta).$

Donc

$$\int_0^1 < g(t(a+y+z)) - g(ta), a > dt = \int_0^1 < g(t(a+y)) + g(t(a+z)) - 2g(ta), a > dt$$

et de l'hypothèse 3) il résulte que

$$\int_0^1 < g(t(a+y+z)), y > dt = \int_0^1 < g(t(a+y)), y > dt$$

et

$$\int_0^1 < g(t(a+y+z)), z > dt = \int_0^1 < g(t(a+z)), z > dt$$

En conclusion

$$f_a(y+z) = \int_0^1 \left(< g(t(a+y)), a+y > + < g(t(a+z)), a+z > -2 < g(ta), a >\right) dt$$

$$=f(a+y)+f(a+z)-2f(a)=f_a(y)+f_a(z),$$

ce qui montre le fait que $f_a$ est ortogonal-additif.

Pour l'operateur $f$ nous avons

$$f(y)+f(y) \le f(z)+f(z),\ (\forall)\ y,z \in E,\ 0 \le y \le z$$

Puisque $g$ est monotone – croissante, il résulte que

$$0=g(0) \le g(ty) \le g(tz)\ \ \forall t \in (0,\infty),\ y,z \in E,\ 0 \le y \le z.$$

Donc

$$<g(ty),y> \le <g(tz),z>,\ \forall t \in (0,\infty) \Rightarrow f(y) \le f(z).$$

De la même manière, puisque $0 \ge -y \ge -z$, il résulte que

$$0 = g(0) \ge g(-ty) \ge g(-tz),\ \forall t \in (0,\infty)$$

et donc

$$< g(-ty),-y > \le < g(-tz),z > \ \Rightarrow f(-y) \le f(-z).$$

En conclusion pour $g(0)=0$ nous avons

$$f(y)+f(-y) \le f(z)+f(-z).$$

Si $g(0) \ne 0$ on considère

$$g_0(x)=g(x)-g(0),\ \forall x \in E.$$

et

$$F : E \to \Re,\ F(x) = \int_0^1 < g_0(tx), x > dt,\ x \in E$$

Puisque $g_0(0)=0$, on a

$$F(y)+F(-y) \le F(z)+F(-z),\ \forall y,z \in E,\ 0 \le y \le z.$$

On vérifie

$$F(x)=f(x)-<f(0),x>,\ \forall x \in E$$

et donc

$$f(y)+f(-y) \le f(z)+f(-z),\ \forall y,z \in E,\ 0 \le y \le z.$$

Soient $a \in E,\ y,y' \in E$ telle que $|y| \le |y'|$.

Puisque

$$f_a'(x) = g(a+x),\ \forall x \in E\ \ et\ \ f_a(0) = 0$$

il résulte que

$$f_a(x)=\int_0^1 <g(a+tx),x>dt,\ \forall x\in E$$

$$\Rightarrow f_a(|y|)+f_a(-|y|)\le f_a(|y'|)+f_a(-|y'|),\ \forall |y|\le |y'| \Rightarrow$$

$f$ est $(s)$ – monotone – croissante.

Dans cet théorème on donné une caractérisation de $(o)$ – convexité en termes de monotonie de gradient.

## THÉORÈME 3

Soit $X$ un espace Banach qui est aussi un espace linéaire $\sigma$-reticulé et $\mathbf{E}\subset\mathbf{X}$ un sous-espace linéaire reticulé. Si $\mathbf{g}:\mathbf{E}\to\mathcal{L}(\mathbf{X},\Re)$ est gradient continu pour $\mathbf{f}:\mathbf{E}\to\Re$, alors les affirmations suivantes sont équivalentes:

a) $f$ est (o)-convexe et si $a\in E$, $f_a$ est ortogonale-additife;

b) l'opérateur $g$ satisfait les conditions:

  i) pour $a\in E$, $g_a$ est ortogonal-additif;

  ii) $g$ est monotone-croissante;

  iii) si $y,z\in E$ et $y\perp z$, alors $<g(x+y),z>=<g(x),z>\ (\forall)\ x\in E$.

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# THE HERMITE-HADAMARD INEQUALITY FOR CONVEX FUNCTIONS OF A VECTOR VARIABLE

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ABSTRACT. The Hermite-Hadamard inequality is discussed in the light of Choquet's theory.

It is well known that every convex function $f : [a, b] \to \mathbb{R}$ can be modified at the endpoints to become convex and continuous. An immediate consequence of this remark is the integrability of $f$. The *mean value* of $f$,

$$M(f) = \frac{1}{b-a} \int_a^b f(x)\, dx,$$

can then be estimated by the *Hermite-Hadamard Inequality*,

$$\text{(HH)} \qquad f\left(\frac{a+b}{2}\right) \leq M(f) \leq \frac{f(a)+f(b)}{2},$$

which follows easily from the midpoint and trapezoidal approximation to the middle term. Moreover, under the presence of continuity, equality occurs (in either side) only for linear functions. An updated account on $(HH)$ are to be found in [2].

What about the case of functions of several variables? A recent paper by S. S. Dragomir [3] (see also [2]) describes the case of balls in $\mathbb{R}^3$, by proving that

$$f(a) \leq \frac{1}{Vol\, \overline{B}_R(a)} \iiint_{\overline{B}_R(a)} f(x)\, dV \leq \frac{1}{Area\, S_R(a)} \iint_{S_R(a)} f(x)\, dS$$

for every continuous convex function $f : \overline{B}_R(a) \to \mathbb{R}$. However, as we shall show in the sequel, more general results are already available in the existing literature. In fact, the right approach of the entire subject of Hermite-Hadamard type inequalities comes from Choquet's theory, a theory whose highlights were presented by R. R. Phelps in his booklet [5]. For a more advanced material, see the monograph of E. M. Alfsen [1].

The basic observation is that the middle point $(a+b)/2$ represents the barycenter of the given interval $[a, b]$ (with respect to a uniform distribution of mass), while the right hand side of (HH) represents the mean value of $f$ over the set of extreme points of the given interval.

Then the two sides of (HH) follow different routes, with different degrees of generality.

To enter the details, let $K$ be a compact convex subset $K$ of a locally convex Hausdorff space $E$ and suppose there is given a Radon probability measure $\mu$ on $K$

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(which can be thought of as a mass distribution on $K$). The $\mu-$ *barycenter* of $K$ is defined as the unique point $x_\mu$ of $K$ such that

$$x'(x_\mu) = \int_K x'(x)\, d\mu(x) \tag{B}$$

for every continuous linear functional $x'$ on $E$; see [5], Proposition 1.1. When $E$ is the Euclidean $n-$dimensional space, the normed and the weak convergence are the same, so that

$$x_\mu = \int_K x\, d\mu(x)$$

i.e., the barycenter coincides with the moment of first order of $\mu$.

An immediate consequence of (B) is the validity of the inequality

$$f(x_\mu) \leq \int_K f(x)\, d\mu(x)$$

for every continuous convex function $f : K \to \mathbb{R}$, a fact which extends the left part of the classical Hermite-Hadamard inequality. For details, see the remark before Lemma 4.1 in [5]. Another remark is the following monotonicity property (noticed by S. S. Dragomir [3] in a particular case):

**1. Proposition**. *Under the above hypothesis, the function*

$$M(t) = \int_K f(tx + (1-t)x_\mu)\ d\mu(x)$$

*is convex and nondecreasing on* $[0, 1]$.

When $E = \mathbb{R}^n$ and $\mu$ is the Lebesgue measure, the value of $M$ at $t$ equals the mean of $f|K_t$, where $K_t$ denotes the image of $K$ through the mapping $x \to tx + (1-t)x_\mu$, i.e.,

$$M(t) = \frac{1}{\mu(K_t)} \int_{K_t} f(x)\, d\mu(x).$$

Proposition 1 tells us that shrinking $K$ to $x_\mu$, via the sets $K_t$, the mean of $f|K_t$ decreases to $f(x_\mu)$. The proof will need the following approximation argument, which was shown to us by Prof. Gheorghe Bucur:

**2. Lemma.** *Every Radon probability measure $\mu$ on $K$ is the pointwise limit of a net of discrete Radon probability measures $\mu_\alpha$ on $K$, which have the same barycenter as $\mu$.*

*Proof.* We have to prove that for every $\varepsilon > 0$ and every finite family $f_1, ..., f_n$ of continuous real functions on K there exists a discrete Radon probability measure $\nu$ such that

$$x_\nu = x_\mu \quad \text{and} \quad \sup_{1 \leq k \leq n} |\nu(f_k) - \mu(f_k)| < \varepsilon.$$

As $K$ is compact and convex and the $f_k$'s are continuous, there exists a finite covering $(D_\alpha)_\alpha$ of $K$ by open convex sets such that the oscillation of each of the functions $f_k$ on each set $D_\alpha$ is $< \varepsilon$. Let $(\varphi_\alpha)_\alpha$ be a partition of the unity, subordinated to the covering $(D_\alpha)_\alpha$ and put

$$\nu = \sum\nolimits_\alpha \mu(\varphi_\alpha)\, \varepsilon_{x(\alpha)}$$

where $x(\alpha)$ is the barycenter of the measure $f \to \mu(\varphi_\alpha f)/\mu(\varphi_\alpha)$. As $D_\alpha$ is convex and the support of $\varphi_\alpha$ is included in $D_\alpha$, we have $x(\alpha) \in \overline{D}_\alpha$. On the other hand,

$$\mu(h) = \sum_\alpha \mu(h\varphi_\alpha) = \sum_\alpha \frac{\mu(h\varphi_\alpha)}{\mu(\varphi_\alpha)} \cdot \mu(\varphi_\alpha) = \sum_\alpha h(x(\alpha)) \cdot \mu(\varphi_\alpha) = \nu(h)$$

for every continuous affine function $h : K \to \mathbb{R}$. Consequenly, $\mu$ and $\nu$ have the same barycenter. Finally, for each $k$,

$$\begin{aligned} |\nu(f_k) - \mu(f_k)| &= \left|\sum_\alpha \mu(\varphi_\alpha)\, f_k(x(\alpha)) - \sum_\alpha \mu(\varphi_\alpha\, f_k)\right| \\ &= \left|\sum_\alpha \mu(\varphi_\alpha) \left[f_k(x(\alpha)) - \frac{\mu(\varphi_\alpha\, f_k)}{\mu(\varphi_\alpha)}\right]\right| \\ &\le \varepsilon \cdot \sum_\alpha \mu(\varphi_\alpha) = \varepsilon. \quad \blacksquare \end{aligned}$$

*Proof of Proposition* 1. A straightforward computation shows that $M(t)$ is convex and $M(t) \le M(1)$. Then, assuming the inequality $M(0) \le M(t)$, from the convexity of $M(t)$ we infer

$$\frac{M(t) - M(s)}{t - s} \ge \frac{M(s) - M(0)}{s} \ge 0$$

for $0 \le s < t \le 1$ i.e., $M(t)$ is nondecreasing. To end the proof, it remains to show that $M(t) \ge M(0) = f(x_\mu)$. For, choose a net $(\mu_\alpha)_\alpha$ of discrete Radon probability measures on $K$, as in Lemma 2 above. Clearly,

$$f(x_\mu) \le \int_K f(tx + (1-t)x_\mu)\, d\mu_\alpha(x) \quad \text{for all } \alpha$$

and thus the desired conclusion follows by passing to the limit over $\alpha$. $\blacksquare$

The extension of the right hand inequality in (HH) is a bit more subtle and makes the object of Choquet's theory, briefly summarized in the sequel. Given two Radon probability measures $\mu$ and $\lambda$ on $K$, we say that $\mu$ is *majorized* by $\lambda$ (i.e., $\mu \prec \lambda$) if

$$\int_K f(x)\, d\mu(x) \le \int_K f(x)\, d\lambda(x)$$

for every continuous convex function $f : K \to \mathbb{R}$. As noticed in [5], $\prec$ is a partial ordering on the set of all Radon probability measures on $K$.

**3. The Choquet Theorem** ([5], ch. 3). *Let $\mu$ be a Radon probability measure on a metrizable compact convex subset $K$ of a locally convex Hausdorff space $E$. Then there exists a maximal Radon probability measure $\lambda \succ \mu$ such that the following two conditions are verified*:

*i) The barycenter of $K$ with respect to $\lambda$ and $\mu$ is the same*;

*ii) The set $\mathcal{E}xt\, K$ of all extremal points of $K$ is a $G_\delta$−subset of $K$ and $\lambda$ is concentrated on $\mathcal{E}xt\, K$* (i.e., $\lambda(K \setminus \mathcal{E}xt\, K) = 0$).

Under the hypotheses of the above result we get

$$\text{(Ch)} \qquad f(x_\mu) \le \int_K f(x)\, d\mu(x) \le \int_{\mathcal{E}xt\, K} f(x)\, d\lambda(x)$$

for every continuous convex function $f : K \to \mathbb{R}$, a fact which represents a full extension of (HH) in the case of *metrizable* compact convex sets. Notice that the right part of (Ch) reflects the *maximum principle* for convex functions.

In general, $\lambda$ is not unique, except for the case of simplices; see [5], ch. 9.

Another useful remark is that every Radon probability measure $\lambda$, concentrated on $\mathcal{E}xt\,K$, for which (Ch) holds, is maximal. Cf. [5], Corollary 9.8.

According to the above discussion, if $K = [a,b]$, then necessarily $\lambda$ is a convex combination of the Dirac measures $\varepsilon_a$ and $\varepsilon_b$, say $\lambda = (1-\alpha)\varepsilon_a + \alpha\varepsilon_b$. This remark yields Fink's Hermite-Hadamard type inequality [4] in the case of probability measures:

$$\text{(F)} \qquad \int_a^b f(x)\,d\mu(x) \le \frac{b - x_\mu}{b-a}\cdot f(a) + \frac{x_\mu - a}{b-a}\cdot f(b)$$

*for every continuous convex functions* $f : [a,b] \to \mathbb{R}$ *and every Radon probability measure* $\mu$ *on* $[a,b]$; as usually, $x_\mu$ denotes the barycenter of $\mu$, i.e, $x_\mu = \int_a^b x\,d\mu(x)$. In fact, checking

$$\int_a^b f(x)\,d\mu(x) \le (1-\alpha)\cdot f(a) + \alpha\cdot f(b)$$

for $f(x) = (x-a)/(b-a)$ and $f(x) = (b-x)/(b-a)$ we obtain

$$\alpha \ge \frac{x_\mu - a}{b-a} \quad \text{and respectively } 1-\alpha \ge \frac{b-x_\mu}{b-a}$$

i.e., $\alpha = (x_\mu - a)/(b-a)$.

The argument above can be extended easily for all continuous convex functions defined on $n$–dimensional simplices $K = [A_0, A_1, ..., A_n]$ in $\mathbb{R}^n$. Then the corresponding analogue of (F) for Radon probability measures $\mu$ *on* $K$ will read as

$$f(X_\mu) \le \int_K f(x)\,d\mu \le \sum_{k=0}^{n} Vol_n\left([A_0, A_1, .., .\widehat{A_k}, ..., A_n]\cdot f(A_k)\right);$$

here $X_\mu$ denotes the barycenter of $\mu$, and $[A_0, A_1, .., .\widehat{A_k}, ..., A_n]$ denotes the sub-simplex obtained by replacing $A_k$ by $X_\mu$; this is the sub-simplex opposite to $A_k$, when adding $X_\mu$ as a new vertex. $Vol_n$ represents the Lebesgue measure in $\mathbb{R}^n$.

In the case of closed balls $K = \overline{B}_R(a)$ in $\mathbb{R}^3$, $\mathcal{E}xt\,K$ coincides with the sphere $S_R(a)$; the paper by Dragomir [3] illustrates the aforementioned theorem of Choquet in the case where $\mu$ is the normalized Lebesgue measure on $\overline{B}_R(a)$. His argument, based on Calculus, avoids Choquet's theory, but it cannot be extended to arbitrary compact convex sets and arbitrary Radon probability measures on them.

The Choquet theory is today a well established subject in Mathematics, with many extensions and ramifications, and Theorem 3 above is just the beginning of the story. The reader will find much fun formulating many other results in the Choquet theory as Hermite-Hadamard type inequalities.

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# Induced representations of hypergroups

*Liliana Pavel*

First we shortly present the main directions of the study of induced representations of hypergroups so far, and we describe in which way the results in this topic have been naturaly inspired by the analogous results on locally compact groups. We illustrate these ideas relating positive definite measures on hypergroups to induced representations of hypergroups.

Although great progress has been made since the beginning of systematic studies of representations of hypergroups, the concept of induced representations has not been developed in the general case of an arbitrary closed subhypergroup H of $K$ up to Hermann [7]. The most general result prior to this paper was the one of Hauenschild, Kaniuth, Kumar [6]: the authors laid the foundations to induce a representation from a closed *subgroup* (that means a subgroup of the maximal group $(G(K) = \{ x \in K \mid \delta_x * \delta_{x^\vee} = \delta_{x^\vee} * \delta_x = \delta_e \})$ of $K$. They translated the classical induction procedure basically invented by G. Mackey to the case of hypergroups, when $H$ is a closed subgroup of $K$. The main tool that allows this translation and supports the whole computation is: $\forall x \in K,\ t \in G(K)$, there exists an unique $y$ (denoted by $xt$) such that $\delta_x * \delta_t$ =

this way is a Hermitian $A$-module, which is called the Hermitian $A$-module obtained by inducing $V$ up to $A$ via $P$.

Actually, if $H$ is a closed subgroup of a locally compact group $G$, then the $C^*$-algebra $C^*(H)$ of $H$ is not o subalgebra of $C^*(G)$, but rather acts as an algebra of right centralizers on $C^*(G)$. Also the natural candidate for a conditional expectation from $C^*(G)$ to $C^*(H)$ is not continuous, or everywhere defined. For these reasons, Rieffel generalized the definition of a conditional expectation and surpassed some technical obstacles, including the Mackey's construction as a special case of his construction for $C^*$-algebras. The obstacles are the same when $H$ is a closed subhypergroup of the hypergroup $K$.

In his work, [7], Hermann has translated the way in which Rieffel surpassed the above difficulties, directly, when $H$ is a closed subhypergroup of the hypergroup $K$, describing the Rieffel's inducing process for the $C^*$-algebras, $C^*(H)$, $C^*(K)$.

Roughly speaking, hypergroups are locally compact spaces, whose regular complex valued Borel measures form an algebra which has properties similar to the convolution algebra $(M(G),*)$ of a locally compact group. For basic notations and references, one can consult the paper of Jewett, [8]. Let $K$ be a hypergroup. We shall denote by $C_c(K)$ the space of all complex valued continuous functions with compact support on $K$ and by $M(K)$ the bounded regular Borel measures on $K$. Furthermore, all hyperhroups occuring in this paper are supposed to admit a (left) Haar measure, $m_K$. Its modular function is denoted by $\Delta_K$. We mention that it is still unknown if an arbitrary hypergroup admits a left Haar measure, but all the known examples such as commutative and central hypergroups [6] do; in addition, in [15] it is proved that each subhypergroup of a hypergroup having a left Haar measure, also admits a left Haar measure. With this assumption, one can define the convolution algebra, $L^1(K)$. Further, as the left regular representation of $K$ is faithful, we can embed $L^1(K)$, as well as $C_c(K)$ in its enveloping $C^*$-algebra, $C^*(K)$. By analogy to the

$=\delta_y$. Consequently, any technique based on this kind of computation (Mackey) fails when $H$ is only a *subhypergroup*.

Studying the paper of Rieffel [12], who invented an abstract inducing process from which the Mackey construction can be obtained as a particular case, working in the groupal algebras of a locally compact group, Hermann has translated the way of obtaining the Mackey construction from the abstract Rieffel inducing process, working in a proper, carefull way in the hypergroupal algebras. We notice that there are a lot of analogies between the natural algebras associated to a locally compact group and to a hypergroup possesing a Haar measure.

Rieffel's inducing process for $C^*$-algebras is rougly as follows. Let $A$ be a $C^*$-algebra, let $B$ be a subalgebra of $A$, and let $V$ be a Hermitian $B$-module, that is the Hilbert space of a nondegenerate $*$-representation of $B$. It is considered the algebraic tensor product $A\otimes_B V$ and it is tried to answer the question how to equip this $A$-module with an inner product in such a way such that to obtain a nondegenerate $*$-representation of A. An analysis of this question shows that in general there are many different ways of doing this, in contrast to Mackey's theory for locally compact groups where there seems to be essentially only one natural choice of inner-product. This difference is explained by the fact that in the case of locally compact group $G$ and a closed subgroup $H$, an additional piece of structure is present, namely the restriction map from functions on $G$ to $H$. Rieffel noticed that, roughly speaking, this map is a conditional expectation, where for a $C^*$-algebra $A$ and a subalgebra $B$, a conditional expectation is a positive projection, $P$, of $A$ onto $B$, having the property $P(ab)=P(a)b$, for all $a\in A$, $b\in B$. Once a conditional expectation has been chosen, there is a canonical choice of a preinner-product on $A\otimes_B V$, whose value on elementary tensors is given by: $\langle a_1\otimes v_1, a_2\otimes v_2\rangle = \langle P(a^* a_l)v_1, v\rangle_V$. This definition, is very closed to that used in the Gelfand-Naimark-Segal construction of a representation from a positive linear functional. The Hilbert space obtained in case of a locally compact group, the irreducible representations of $K$ are in one to one correspondence with the non-degenerate (irreducible) representations of $L^1(K)$, and hence $C^*(K)$. Corresponding representations of the objects $K$, $L^1(K)$, $C^*(K)$ will be denoted by the same letter.

Let $H$ be a closed subhypergroup of $K$. Hermann [7], considered "." the action of $C_c(H)$ in $C_c(K)$ given by

$$(f,\varphi)\mapsto f\cdot\varphi = f*\left(\frac{\Delta_{K|H}}{\Delta_H}\right)^{\frac{1}{2}}\varphi,\quad C_c(K)\times C_c(H)\to C_c(K).$$

Consequently,

$$(f\cdot\varphi)(x)=\int_H f(x*t^{\vee})((\Delta_H\Delta_K)(t))^{-\frac{1}{2}}\varphi(t)dm_H(t).$$

As observed in [7, Lemma1]

$$(f*\varphi)\cdot\varphi = f*(g\cdot\varphi),\ \forall f,g\in C_c(K),\ \varphi\in C_c(H).$$

He also considered the map $P\colon C_c(H)\to C_c(K),\ P=\left(\frac{\Delta_{K|H}}{\Delta_H}\right)^{\frac{1}{2}} f|_H$.

With Lemma 2 and 3 [7], the map $P$ is a *-linear, generalized conditional expectation, $P(f)^* = P(f^*)$ and $P(f)*\varphi) = P(f\cdot\varphi),\ \forall f\in C_c(K),\ \varphi\in C_c(H)$.

Contrary to the group case, it is not possible to induce each representation of an arbitrary closed subhypergroup $H$ to $K$, because the Rieffel process requires the positivity of the preinner product and the norm-conrinuity of the representation that are automatically fulfilled working with the groupal algebras. So, the central notion of the Hermann's paper is the *inducibility* of a representation of $H$. Bearing in mind the abstract Rieffel inducing process he defined the inducibility as follows:

**Definition** *(Hermann). Let $\rho$ be a representation of $H$, $\rho$ is said to be inducible (from H) to K if the following two conditions hold:*

*(i) $\rho$ is P-positive, i.e.*

$$\langle\rho(P((f^**f))v,v\rangle\geq 0,\ \forall f\in C_c(K),\ v\in H_\rho;$$

*(ii) $\rho$ is P-bounded, i.e.*

$$\langle \rho(P((f*g)^* * (f*g))v, v\rangle \le \|f\|^2_{C^*(K)} \langle \rho(P((g^* * g))v, v\rangle, \ \forall f,g \in C_c(K),\ v \in H_\rho$$

If (i) and (ii) in the definition above are satisfied one can define a preinner product on the space $V = A \otimes H_\rho$ (algebraic tensor product) by setting on $V \times V$,

$$\langle f \otimes v, g \otimes w\rangle \mapsto \langle \rho(P(g^* * f)v), w\rangle_{H_\rho}.$$

It is a part of the Rieffel theory that this preinner product is well defined. If we set $N = \{x \in V \mid \langle x,x\rangle_{H_\rho} = 0\}$, $\langle\cdot,\cdot\rangle_{H_\rho}$ becomes an inner product on $V/N$. Finally, if $H_{ind\rho}$ is the completion of $V/N$ with respect to this inner product, we get the so-called induced Hilbert space on which the induced representation $ind_{B\uparrow A}\, \rho$ (where $A = C_c(K)$, $B = C_c(H)$) is defined by

$$ind_{B\uparrow A}\rho(f)(g \otimes v) = (f * g) \otimes v.$$

It is clear that every represntation of the subgroup of $K$ is inducible to $K$, so the definition includes the most general case known so far, [6].

In his elegant note appeared in 1963 [2], Blattner shows that an alternate definition of Mackey's induced representations (for groups) can be given in terms of lifting positive definite measures from subgroups. He stated the next result.

**Theorem** *(Blattner). Let G be a locally compact group, H a closed subgroup of G. If $\mu$ is a positive dfinite measure on H, we let $\tilde{\mu}$ be the measure on G obtained by inflating* $\left(\frac{\Delta_{K|H}}{\Delta_H}\right)^{\frac{1}{2}}\mu$, $\tilde{\mu}(f) = \mu\left(\left(\frac{\Delta_{K|H}}{\Delta_H}\right)^{\frac{1}{2}} f|_H\right)$, $f \in C_c(G)$. *Then $\tilde{\mu}$ is positive definite and $T^{(\tilde{\mu})}$ is equivalent to $ind_{H\uparrow K} T^{(\mu)}$.*

(The meanning of $T^{(\mu)}$ is the standard one, $T_g^{(\mu)}([f]) = [g * f]$, $\forall f,g \in C_c(G)$.)

Looking at all these prior works, some classical, concerning the induction process on locally compact groups ([9], [2], [12]) and some connected to induced representations of hypergroups ([7], [6]), we were also tempted to adapt Blattner's idea to induce representations of subhypergroups to hypergroups. In addition, a nice, ambitious and promising program could be initiated: using this idea one might hope to answer questions on imprimitivity for representations of hypergroups, so to get an answer to the difficult question: which representations of $K$ are induced from representations of a given subhypergroup $H$? (Mackey [9], Theorem 6.6). The next results might be considered the first steps to get close to this objective.

First stage in developping Blattner's idea on hypergroups is to study positive definite measures on hypergroups, surpassing the relative poverty of prior works on this subject. There is no hope for a positive definite measure on $K$ (in the usual sense $\int_K f^* * f d\mu \geq 0,\ \forall f \in C_c(K)$) to give rise to a proper representation of the hypergroup, so certainly one has to restrict the class in a proper way. Which should be the measure's properties (except the positivity) in order to ensure that $T^{(\mu)}$ is a representation of the hypergroup?

The *(B)*-boundedness condition seems at the first glance unnatural, and not very confortable to handle with: *a positive definite measure is called (B)-positive definite if*

$$\int_K f^* * g^* * g * f d\mu \leq \|g\|_1^2 \int_K f^* * f d\mu, \forall f, g \in C_c(K).$$

Fortunately, a carefull analysis shows that this condition is automatically fulfilled for several nice cases such as positive definite measures generated by bounded positive definite functions ad postive definite measures (bounded) on a commutative hypergroup. Moreover, one can link this condition with former work on subjects connected to positive definite measures on hypergroups such as [3], [5].

As it follows from Hermann's work [7], not every representation of a closed subhypergroup $H$ can in general be induced to $K$. Accordingly, in order to obtain an analogue to Blattner's description of the induced representations in terms of positive definite measures, we need for the measure $\mu$ a property which is *a substitute* for Hermann's *inducibility*: the natural condition is the one from the definition of *inducible measure*.

**Definition.** *A (B)-positive measure $\mu$ on H is said to be inducible from H to K if $\tilde{\mu}$ is B-positive definte on K.*

For $\mu$ inducible from $H$ to $K$, $\tilde{\mu}$ is denoted by $ind_{H\uparrow K}\,\mu$.

With this definition we naturally translate the Blattner's procedure to hypergroups.The problem that must be solved further is if this construction concords to the prior work of Hermann [7]. We gave a complete, definitively positive answer to this question with the next theorem, our main result:

**Theorem.** *Assume H is a closed subhypergroup of the hypergroup K, and let $\mu$ be a (B)- positive definite measure on H. Then, $\mu$ is inducible from H to K if and only if the representation $T^{(\mu)}$ is inducible from H to K (in the sense of [7]). In this case, $T^{(ind_{H\uparrow K}\mu)}$ is equivalent to $ind_{H\uparrow K}T^{(\mu)}$.*

We can add that this procedure includes the Blattner's description of induced representations in terms of positive definite functions, obtained in [6] for the particular case of $H$ a closed subgroup.

We also obtained the Theorem on Induction in stages, whose proof in this approach is very attractive and natural.

Finally, we notice that this induction procedure on hypergroups, combined with Hermann's results [7], can be used to show in certain cases that a *(B)*-positive definite measures on a subhypergroup $H$ of $K$ can be "extended" to a positive definite measure on $K$. For example, if $H$ is a compact subhypergroup of $K$ or $H$ is a subgroup of $K$, then by [7], each representation $H$ can be induced to $K$. In view of our theorem, each *(B)*-positive definite measure on $H$ is inducible from $H$ to $K$.

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# SOME APPROXIMATION RESULTS IN LOCALLY CONVEX LATTICES

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Let $X$ be a compact Hausdorff space and let $F$ be a vector subspace of $C(X)$. In the approximation theory one can define two classes of subsets of $X$, with respect to the vector subspace $F$, namely: antisymmetric sets and frontal sets. In this paper we firstly remind the main properties of these classes then we show how these concepts and their properties can be generalized to the closed order ideals in locally convex lattices.Typical examples of such lattices are the weighted spaces.

## 1. ANTISYMMETRIC SETS

Let $X$ be a compact Hausdorff space and let $C(X)$ be the space of all continuous complex valued functions on $X$, equipped with the topology of uniform convergence.

We denote by $A^0$ the polar set of any subset $A$ of $C(X)$. Also, for every subset $K$ of $X$, we denote by $\chi_K$ the characteristic function of $K$ and by $I_K = \{f \in C(X); \quad f|K = 0\}$.

**Definition 1.1.** *Let $F$ be a vector subspace of $C(X)$. A subset $S$ of $X$ is said to be antisymmetric with respect to $F$ ($F$ - antisymmetric) if every $f \in F$ with the properties:*

*a) $f|S$ is real valued*

*b) $fg|S \in F|S$ for any $g \in F$, is constant on $S$.*

**Remark 1.1.** *Let $\mathcal{A}$ be a subalgebra of $C(X)$. A subset $S$ of $X$ is said to be antisymmetric with respect to $\mathcal{A}$ if every $f \in \mathcal{A}$, real-valued on $S$, is constant on $S$. Thus, we reobtain the concept of antisymmetric set introduced by E.Bishop in 1961.*

**Theorem 1.1. (G.Paltineanu 1978)** *Let $F$ be a closed vector subspace of $C(X)$. Then:*

*i) The family $\mathcal{S}$ of all maximal $F$ - antisymmetric subsets of $X$ forms a pairwise disjoint partition of $X$.*

*ii) A function $f \in C(X)$ belongs to $F$ iff $f|S \in \overline{F|S}$, for any $S \in \mathcal{S}$.*

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*iii)* $F|S$ *is closed in* $C(S)$ *for any* $S \in \mathcal{S}$.

This result generalizes the well known Bishop's approzimation theorem.

## 2. FRONTAL SETS

The concept of frontal set with respect to a vector subspace of $C(X)$ was introduced by Alain Bernard in 1967 as a generalization of the concept of intersection of peak sets with respect to a subalgebra of $C(X)$.

**Definition 2.1.** *A closed subset* $K$ *of* $X$ *is a frontal set with respect to the vector subspace* $F$ *of* $C(X)$, *if for any* $f \in F$, *any neighborhood* $V$ *of* $K$, *any* $\varepsilon > 0$ *and any* $\eta > 0$, *there is a* $\overline{f} \in F$ *such that:*

$$\overline{f}\,|K = f\,|K\,;\quad \|\overline{f}\|_X \leq \|f\|_K + \eta,\quad \|\overline{f}\|_{X\setminus V} < \varepsilon.$$

**Remark 2.1.** *Let* $\mathcal{A}$ *be a closed subalgebra of* $C(X)$ *containing the constants and* $K$ *be a closed subset of* $X$. *Then,* $K$ *is a frontal set with respect to* $\mathcal{A}$, *iff* $K$ *is an intersection of the peak sets with respect to* $\mathcal{A}$.

The following theorem is a characterization theorem for the frontal sets.

**Theorem 2.1.** *Let* $X$ *be a compact Hausdorff space,* $F$ *a vector subspace of* $C(X)$ *and* $K$ *a closed subset of* $X$ *such that* $F/F \cap I_K$ *is complete. Then, the following assertions are equivalent:*

*(i)* $K$ *is a frontal set with respect to* $F$.

*(ii)* $\chi_K F^0 \subset F^0$.

*(iii) For every* $f \in F$, *and every* $g$ *continuous nonnegative function on* $X$ *such that* $|f(x)| \leq g(x)$ *for* $x \in K$, *and every* $\varepsilon > 0$ *there exists* $\overline{f} \in F$ *with the properties:*

$$\overline{f}\,|K = f\,|K \quad \text{and} \quad |\overline{f}(x)| < g(x) + \varepsilon \quad \text{for} \quad x \in X.$$

**Theorem 2.2.** *The family of all* $F$ *- frontal subsets of* $X$ *is closed with respect to the finite reunions and with respect to any intersection.*

**Theorem 2.3.** *Every maximal* $F$ *- antisymmetric sets is a* $F$ *- frontal set.*

## 3. FRONTAL IDEALS

Our generalization of the concept of a frontal ideal is motivated by Theorem 2.1.

Let $E$ be a real, metrizable, locally convex, locally solid vector lattice and let $\mathcal{V}_0$ be a basis of open, convex, solid neighborhoods of the origin.

**Definition 3.1.** *A closed ideal* $I$ *of* $E$ *is said to be a* $\mathcal{V}_0$ *- frontal ideal with respect to the vector subspace* $F$ *of* $E$ *if for any* $x \in F$ *and* *any $y \in E_+$ with the property $(|x| - y)_+ \in I$ and for any $V \in \mathcal{V}_0$ there exists a $\overline{x} \in F$ such that $x - \overline{x} \in I$ and $(|\overline{x}| - y)_+ \in V$.*

**Remark 3.1.** *If $E = C(X)$, then a closed subset $K$ of $X$ is a frontal set with respect to the vector subspace $F$ of $C(X)$ iff the closed ideal $I_K$ is a frontal ideal in the sense of the Definition 3.1.*

Indeed, it is sufficient to remark that $(|f| - g)_+ \in I_K$ iff $|f(x)| \leq g(x)$, $x \in K$ and that $\|(|\overline{f}| - g)_+\| < \varepsilon$ iff $|f(x)| < g(x) + \varepsilon$ for $x \in X$.

**Theorem 3.1. (G.Paltineanu, D.T.Vuza 1996)** *Let $E$ be a real metrizable, locally convex, locally solid lattice, $I$ a closed ideal of $E$, $F$ a vector subspace of $E$ and $\mathcal{V}_0$ a basis of the origin, consisting of open, convex, solid neighborhoods. If $F/F \cap I$ is complete, then $I$ is a $\mathcal{V}_0$ - frontal ideal with respect to $F$ iff $P_I(F^0) \subset F^0$, where by $P_I$ it was denoted the associated projection $E' \to I^0$.*

Further, we shall denote by $\mathcal{F}_F(E)$ the family of all closed ideals of $E$, $\mathcal{V}_0$ - frontal with respect to $F$.

**Theorem 3.2.** *The family $\mathcal{F}_F(E)$ has the properties:*

*(i) If $(I_\alpha) \subset \mathcal{F}_F(E)$ then $\overline{\sum_\alpha I\alpha} \in \mathcal{F}_F(E)$*

*(ii) If $I, J \subset \mathcal{F}_F(E)$ then $I \cap J \in \mathcal{F}_F(E)$*

*(iii) If $(I_\alpha) \subset \mathcal{F}_F(E)$ and the band generated by $\bigcup_\alpha I\alpha^0$ is $\sigma(E', E)$ - closed, then $\bigcap_\alpha I\alpha \in \mathcal{F}_F(E)$*

(iv) *Let $I$ and $J$ be two closed ideal of $E$ such that $I \subset J$. If $I \in \mathcal{F}_F(E)$ and $F/F \cap I$ is complete, then $J \in \mathcal{F}_F(E)$ iff $J/I$ is frontal ideal of $E/I$ with respect to $F/I$.*

The following result is a generalization of a theorem of a Bernard concerning the frontal set with respect to a closed subspace $F$ of $C(X)$. Given a continuous seminorm $p$ on $E$, we associate to it the quotient seminorm:

$$P_I(x) = \inf \{p(x+u); u \in I\}, \ x \in E.$$

**Theorem 3.3. (C.Niculescu, G.Paltineanu, D.T.Vuza, 2000)** *Let $F$ be a complete vector subspace of $E$, $\mathcal{V}_0$ a basis of convex and solid neighborhoods of the origin, $I$ a $\mathcal{V}_0$-frontal ideal with respect to $F$, $x \in F$, $y \in E_+$ such that $(|x| - y)_+ \in I$, $V \in \mathcal{V}_0$ and $p$ a continuous of $(AM)$-type seminorm on $E$ such that $p_I(x) > 0$. Then, there exists a $\overline{x} \in F$ with the properties:*

$$\overline{x} - x \in I, \ (|\overline{x}| - y)_+ \in V \text{ and } p(\overline{x}) = p_I(x)$$

**Theorem 3.4.** *Let $F$ be a vector subspace of $E$, let $I$ be a $\mathcal{V}_0$-frontal ideal with respect to $F$, $x \in F$, $J$ a closed ideal of $E$ such that*

$x \in \overline{I+J}$, *p a continuous and solid seminorm on E and* $\varepsilon > 0$. *Then there exists a* $\overline{x} \in F$ *with the properties:*

$$\overline{x} - x \in I,\ p(\overline{x}) \leqslant p_I(x) + \varepsilon \text{ and } p_J(\overline{x}) \leqslant \varepsilon$$

*If F is complete, p is a (AM)-type seminorm and* $p_I(x) > 0$ *then* $p(\overline{x}) = p_I(x)$.

## 4. ANTISYMMETRIC IDEALS

Let now $E$ be a real, locally convex, locally solid vector lattice of (AM)-type. The center $Z(E)$ of $E$ is the algebra of order bounded endomorphisms $U \in L(E,E)$, i.e those $U$ for which there exists $\lambda > 0$ such that $|U(x)| \leq \lambda |x|$ for all $x \in E$. We define the real part of the center by $\operatorname{Re} Z(E) = Z(E)_+ - Z(E)_+$.

**Definition 4.1.** *A closed ideal I of E is said to be antisymmetric with respect to the vector subspace F of E if for any* $U \in ReZ(E/I)$ *with the property* $U[\pi_I(F)] \subset \pi_I(F)$ *it follows that there exists* $a \in R$ *such that* $U = a\mathbf{1}_{E/I}$, *where* $\mathbf{1}_{E/I}$ *is the identity operator on* $E/I$.

We shall denote by $\mathcal{A}_F(E)$ the set of all closed ideals of $E$, antisymmetric with respect to $F$.

**Remark 4.1.** *If* $E = C(X)$, *then a closed subset K of X is an antisymmetric set with respect to F iff the closed ideal* $I_K$ *is antisymmetric with respect to F in the sense of Definition 4.1.*

It is sufficient to observe that $Z(C(X)) = C(X)$ and that $\pi_{I_K}(f) = f\,|K$ for every $f \in C(X)$.

**Theorem 4.1. (G.Paltineanu, D.T.Vuza ).** *The family* $\mathcal{A}_F(E)$ *has the properties:*

*(i) If* $(I\alpha) \subset \mathcal{A}_F(E)$ and $J = \sum_\alpha I\alpha \neq E$ *then* $I = \bigcap_\alpha I\alpha \in \mathcal{A}_F(E)$.

*(ii) Every* $I \in \mathcal{A}_F(E)$ *contains a unique minimal ideal* $I_0 \in \mathcal{A}_F(E)$.

*(iii) If the family of all continuous lattice homomorphisms* $h : E \to \mathbf{R}$ *separates the points of E, then the intersection* $\cap \left\{I; I \in \widetilde{\mathcal{A}}_F(E)\right\} = \{0\}$, *where* $\widetilde{\mathcal{A}}_F(E)$ *denoted the family of all minimal antisymmetric F- ideals of E.*

*(iv) Let* $\mathcal{I} = \{I_1, I_2, ..., I_m\} \subset \widetilde{\mathcal{A}}_F(E)$ *and* $\mathcal{J} = \{J_1, J_2, ..., J_n\} \subset \widetilde{\mathcal{A}}_F(E)$ *such that* $I_K \cap J_l = \{0\}$ *for every l. Then* $E = \bigcap_{K=1}^{m} I_K + \bigcap_{l=1}^{n} J_l$.

*(v) If F is complete, then* $\widetilde{\mathcal{A}}_F(E) \subset \mathcal{F}_F(E)$.

The following Lemma generalized de Branges' Lemma.

**Lemma 4.1.** *Let F be a vector subspace of E and V a convex, solid neighbourhood of the origin, which is also sublattice. If* $f \in Ext\{V^0 \cap F^0\}$ *and* $I = \{x \in E;\ |f|(|x|) = 0\}$, *then* $I \in \mathcal{A}_F(E)$.

The main result concerning to antisymmetric ideals is the following Bishop's type approximation theorem.

**Theorem 4.2.** *Let $E$ be a real metrizable l.c. of (AM)-type and let $F \subset E$ be a vector subspace. Then for any $x \in E$ we have:*

$$x \in \overline{F} \text{ iff } \pi_I(x) \in \overline{\pi_I(F)} \text{ for every } I \in \widetilde{\mathcal{A}}_F(E)$$

**Remark 4.1.** *Theorem 4.2 generalizes Theorem 1.1.*

**Corollary 4.2.** *If $F$ is complete and $I \in \widetilde{\mathcal{A}}_F(E)$, then $\pi_I(F)$ is complete.*

**Theorem 4.3.** Suppose that $F$ is complete. Let $\mathcal{I}_1, \mathcal{I}_2$ be two finite and disjoint subsets of $\widetilde{\mathcal{A}}_F(E)$ (i.e. $I \cap J = \{0\}$ for any $I \in \mathcal{I}_1$ and any $J \in \mathcal{I}_2$), and let $I_1 = \cap\{I; I \in \mathcal{I}_1\}$ and $I_2 = \cap\{I; I \in \mathcal{I}_2\}$. Then, for every $x \in F$, every continuous seminorm $p$ on $E$ and every $\varepsilon > 0$, there is a $\overline{x} \in F$ such that:

$$\overline{x} - x \in I_1,\ p(\overline{x}) \leq p_{I_1}(x) + \varepsilon \text{ and } p_{I_2}(x) \leq \varepsilon$$

If in addition $p$ is an (AM)-type seminorm and $p_{I_1}(x) > 0$, then $p(\overline{x}) = p_{I_1}(x)$.

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# LOCALLY BOUNDED SEMIGROUPS

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After a short introduction we have introduced the notion of locally bounded semigroups. At first, we consider locally bounded semigroups on projective limits of normed spaces and a characterization in this context is given.

Secondly we investigate locally bounded semigroups on locally convex spaces and a characterization in terms of projective families of semigroups acting on Banach spaces is given.

## 1. INTRODUCTION

**Definition 1.1.** *Let* $(I, \leq)$ *be an ordered set and* $(\mathscr{X}_\alpha)_{\alpha\in I}$ *a family of topological vector spaces. Suppose that for any* $\alpha, \beta \in I$, $\alpha \leq \beta$ *there exists a continuous linear mapping* $f_{\alpha\beta} : \mathscr{X}_\beta \to \mathscr{X}_\alpha$ *verifying that*

*1. If* $\alpha \leq \beta \leq \delta$ *then* $f_{\alpha\delta} = f_{\alpha\beta} f_{\beta\delta}$.

*2. For any* $\alpha \in I$, $f_{\alpha\alpha} = I_\alpha$. ($I_\alpha$ *is the identical operator on* $\mathscr{X}_\alpha$). *We say that* $(\mathscr{X}_\alpha, f_{\alpha\beta})$ *is a projective family of topological vector spaces. Let us denote by*

$$\lim_{\leftarrow} \mathscr{X}_\alpha = \left\{ y \in \prod_{\alpha\in I} \mathscr{X}_\alpha : \alpha, \beta \in I, \alpha \leq \beta, f_{\alpha\beta}\,\mathrm{pr}_\beta(y) = \mathrm{pr}_\alpha(y) \right\}$$

*the projective limit of* $(\mathscr{X}_\alpha, f_{\alpha\beta})$.

**Example 1.2.** Let $(\mathscr{X}, \mathscr{P})$ be a locally convex space whose topology is given by the family of seminorms $\mathscr{P} = \{p_\alpha : \alpha \in I\}$. Consider on $I$ the following order relation: if $\alpha, \beta \in I$, we say that $\alpha \leq \beta$ iff $p_\alpha \leq p_\beta$. For each $\alpha \in I$, we denote by $J_\alpha = p_\alpha^{-1}(0)$ and $\mathscr{X}_\alpha = \mathscr{X} | J_\alpha$ the quotient space. If $x_\alpha = x + J_\alpha$ we denote by $\|x_\alpha\|_\alpha = p_\alpha(x)$. Thus $\mathscr{X}_\alpha$ becomes a normed space. Let $\alpha, \beta \in I$, $\alpha \leq \beta$ and $f_{\alpha\beta} : \mathscr{X}_\beta \to \mathscr{X}_\alpha$ defined by $f_{\alpha\beta}(x + J_\beta) = x + J_\alpha$. Then $(\mathscr{X}_\alpha, f_{\alpha\beta})$ is a projective family of normed spaces, which is called the projective family associated to $(\mathscr{X}, \mathscr{P})$.

**Remark 1.3.** *Let* $(\mathscr{X}, \mathscr{P})$, $\mathscr{P} = \{p_\alpha : \alpha \in I\}$ *be a Hausdorff locally convex space,* $(\mathscr{X}_\alpha, f_{\alpha\beta})$ *the projective family associated to* $(\mathscr{X}, \mathscr{P})$ *and* $f : \mathscr{X} \to \lim\limits_{\leftarrow} \mathscr{X}_\alpha$ *defined by* $f(x) = (x + J_\alpha)_{\alpha\in I}$. *Then it is clear that* $f$ *realizes an embedding of* $\mathscr{X}$ *into* $\lim\limits_{\leftarrow} \mathscr{X}_\alpha$. *If in addition* $(\mathscr{X}, \mathscr{P})$ *is complete,* $f$ *will be an isomorphism* ([3], *p.*70) *and consequently* $\mathscr{X}$ *and* $\lim\limits_{\leftarrow} \mathscr{X}_\alpha$ *can be identified.*

**Definition 1.4**. *Let* $(\mathscr{X}_\alpha, f_{\alpha\beta})$ *be a projective family of topological vector spaces and* $V_\alpha : D_\alpha \to \mathscr{X}_\alpha$, $\alpha \in I$ *a family of linear operators. We say that* $(V_\alpha)_{\alpha\in I}$ *is a projective family of operators if the following diagram is commutative*

$$\begin{array}{ccc} D_\beta & \xrightarrow{f_{\alpha\beta}} & D_\alpha \\ \downarrow V_\beta & & \downarrow V_\alpha \\ \mathscr{X}_\beta & \xrightarrow{f_{\alpha\beta}} & \mathscr{X}_\alpha \end{array}$$

*In this case the linear operator*

$$V : \left(\prod_{\alpha\in I} D_\alpha\right) \cap \lim_{\leftarrow} \mathscr{X}_\alpha \to \lim_{\leftarrow} \mathscr{X}_\alpha$$

*defined by*

$$V(y) = \left(V_\alpha(\mathrm{pr}_\alpha(y))\right)_{\alpha\in I}$$

*is called the projective limit of* $(V_\alpha)_{\alpha\in I}$ *and it is denoted by* $V = \lim\limits_{\leftarrow} V_\alpha$.

**Definition 1.5.** *Let* $(\mathscr{X}_\alpha, f_{\alpha\beta})$ *be a projective family of topological vector spaces and for each* $\alpha \in I$, $T_\alpha$ *a semigroup of linear operators on* $\mathscr{X}_\alpha$. *We say that* $(T_\alpha)_{\alpha\in I}$ *is a projective family of semigroups if for each* $t \geq 0$, $(T_\alpha(t))_{\alpha\in I}$ *is a projective family of operators. In this case the mapping* $T : [0, \infty) \to L\left(\lim\limits_{\leftarrow} \mathscr{X}_\alpha\right)$ *defined by* $T(t) = \lim\limits_{\leftarrow} T_\alpha(t)$ *is a semigroup and is called the projective limit of* $(T_\alpha)_{\alpha\in I}$. *(For short* $T = \lim\limits_{\leftarrow} T_\alpha$*).*

## 2. LOCALLY BOUNDED SEMIGROUPS

**Definition 2.1.** *Let* $\mathscr{X}$ *be a locally convex space and* $T : [0, \infty) \to L(\mathscr{X})$ *a semigroup. We say that T is locally bounded if there exists a family of seminorms Q which gives the topology of* $\mathscr{X}$, *verifying that: for any* $p \in Q$, *there exist* $M(p) > 0$ *and* $\omega(p) > 0$ *such that* $p(T(t)(x)) \leq M(p)e^{\omega(p)t}p(x)$ *for all* $x \in \mathscr{X}$ *and* $t \geq 0$. *In this context we say that T is Q-locally bounded.*

**Proposition 2.2.** *Let* $\mathscr{X}$ *be a locally convex space,* $\mathscr{P}$ *the family of all continuous seminorms on* $\mathscr{X}$ *and* $S : [0, \infty) \to L(\mathscr{X})$ *a semigroup. Then the following statements are equivalent:*

*1. S is locally bounded.*

*2. For any* $p \in \mathscr{P}$, *there exist* $q \in \mathscr{P}$ *and* $\omega > 0$ *such that* $p(S(t)(x)) \leq e^{\omega t}q(x)$ *for all* $x \in \mathscr{X}$ *and* $t \geq 0$.

*Proof.* If $S$ is locally bounded, from Definition 6 it obviously follows that the second condition is fulfiled. Let us suppose that $S$ fulfils the second condition. Let $p \in \mathscr{P}$, $\omega > 0$ and $q \in \mathscr{P}$ given by the second statement. We can define the following functional

$$u_p : \mathscr{X} \to \mathbf{R}, \text{ by } u_p(x) = \sup_{t\geq 0} e^{-\omega t} p(S(t)(x)).$$

It is clear that $u_p \geq p$ and $u_p \leq q$, which implies that the family $Q = \{u_p : p \in \mathscr{P}\}$ is equivalent to $\mathscr{P}$. Let now $t, s \in [0, \infty)$ and $x \in \mathscr{X}$. Then we have:

$$e^{-\omega(t+s)} p(S(t+s)(x)) \leq u_p(x)$$

and

$$e^{-\omega t}\cdot e^{-\omega s}p\big(S(t)S(s)(x)\big)\le u_p(x)$$

Multiply the last inequality by $e^{\omega s}$ and obtain

$$e^{-\omega t}p\big(S(t)S(s)(x)\big)\le e^{\omega s}u_p(x).$$

By passing to supremum as $t \ge 0$ and by taking into account the definition of $u_p$, we have

$$u_p\big(S(s)(x)\big)\le e^{\omega s}u_p(x).$$

Therefore $S$ is locally bounded. ❑

**Corollary.** *Let $(\mathscr{X},\|,\|)$ be a normed space and $S : [0,\infty) \to L(\mathscr{X})$ a semigroup. Then the following assertions are equivalent:*

*1. S is locally bounded.*

*2. There exist $M > 0$ and $\omega > 0$ such that $\|S(t)\| \le Me^{\omega t}$ for all $t \ge 0$.*

*The proof is a simple consequence of Definition 2.1 and Proposition 2.2.*

Since any Hausdorff locally convex space can be embedded into a projective limit of normed spaces we consider at first, locally bounded semigroups on projective limits of normed spaces.

**Theorem 2.3.** *Let $(I,\le)$ be a ordered set, $(\mathscr{X}_\alpha, f_{\alpha\beta})$ a projective family of normed spaces, $\left(\prod_{\alpha\in I}\mathscr{X}_\alpha, \mathscr{P}\right)$ the product space where $\mathscr{P}=\{p_\alpha : \alpha\in I\}$ and a subspace $Z \subset \varprojlim \mathscr{X}_\alpha$ verifying that $\mathrm{pr}_\alpha(Z)=\mathscr{X}_\alpha$ for all $\alpha \in I$. Let also $S : [0,\infty) \to L(Z)$ be a mapping. Then the following statements are equivalent:*

*1. $S$ is a $\mathscr{P}$-locally bounded semigroup.*

*2. There exists a unique projective family of locally bounded semigroups $(S_\alpha)_{\alpha\in I}$ acting on $\mathscr{X}_\alpha$ such that $S(t)=\left(\varprojlim S_\alpha(t)\right)\Big|_Z$ for all $t \ge 0$.*

*Proof.* $2 \Rightarrow 1$. Let $\alpha \in I$ and $M(\alpha)$ and $\omega(\alpha)$ given by the above Corollary such that $\|S_\alpha(t)\| \le M(\alpha)e^{\omega(\alpha)t}$, for all $t \ge 0$. If $z \in Z$ then $S(t)(z) = \big(S_\alpha(t)(\mathrm{pr}_\alpha(z))\big)_{\alpha\in I}$ and

$$p_\alpha\big(S(t)(z)\big)=\|S_\alpha(t)(\mathrm{pr}_\alpha(z))\|_\alpha \le M(\alpha)e^{\omega(\alpha)t}\|\mathrm{pr}_\alpha(z)\|_\alpha = M(\alpha)e^{\omega(\alpha)t}p_\alpha(z)$$

which means that $S$ is a $\mathscr{P}$-locally bounded semigroup.

$1 \Rightarrow 2$. Let $\alpha \in I$, $t \ge 0$, $x_\alpha \in \mathscr{X}_\alpha$ and $y, z \in Z$ such that $\mathrm{pr}_\alpha(z) = \mathrm{pr}_\alpha(y) = x_\alpha$. Since $S$ is $\mathscr{P}$-locally bounded it follows that $p_\alpha\big(S(t)(y)-S(t)(z)\big)=0$, and hence $\mathrm{pr}_\alpha\big(S(t)(y)\big)= =\mathrm{pr}_\alpha\big(S(t)(z)\big)$. In this context we can define $S_\alpha(t): \mathscr{X}_\alpha \to \mathscr{X}_\alpha$ by $S_\alpha(t)(x_\alpha)=\mathrm{pr}_\alpha\big(S(t)(y)\big)$ where $\mathrm{pr}_\alpha(y)=x_\alpha$. Moreover, $\|S_\alpha(t)(x_\alpha)\|_\alpha = p_\alpha\big(S(t)(y)\big)$.

For $\alpha \in I$ given there exist $M(\alpha) > 0$ and $\omega(\alpha) > 0$ verifying that

$$p_\alpha\big(S(t)(y)\big)\le M(\alpha)e^{\omega(\alpha)t}p_\alpha(y)$$

In conclusion

$$\|S_\alpha(t)(x_\alpha)\|_\alpha \le M(\alpha)e^{\omega(\alpha)t}\|x_\alpha\|_\alpha$$

Let now $t, s \in [0, \infty)$, $\alpha \in I$, $x_\alpha \in \mathscr{X}_\alpha$ and $y \in Z$ so that $\mathrm{pr}_\alpha(y) = x_\alpha$. Then we have

$$S_\alpha(t+s)(x_\alpha) = \mathrm{pr}_\alpha(S(t+s)(y)) = \mathrm{pr}_\alpha(S(t)S(s)(y)).$$

On the other hand,

$$S_\alpha(t)S_\alpha(s)(x_\alpha) = S_\alpha(t)(\mathrm{pr}_\alpha S(s)(y)) = \mathrm{pr}_\alpha(S(t)S(s)(y)).$$

Therefore

$$S_\alpha(t+s)(x_\alpha) = S_\alpha(t)S_\alpha(s)(x_\alpha).$$

In addition we have

$$S_\alpha(0)(x_\alpha) = \mathrm{pr}_\alpha(S(0)(y)) = \mathrm{pr}_\alpha(y) = x_\alpha.$$

Thus, we can assert that $S_\alpha$ is a locally bounded semigroup on $\mathscr{X}_\alpha$, for all $\alpha \in I$.

Let $\alpha, \beta \in I$, $\alpha \leq \beta$, $t \geq 0$, $x_\beta \in \mathscr{X}_\beta$, $y \in Z$ such that $\mathrm{pr}_\beta(y) = x_\beta$. Under these conditions we can write

$$f_{\alpha\beta}S_\beta(t)(x_\beta) = f_{\alpha\beta}\,\mathrm{pr}_\beta S(t)(y) = \mathrm{pr}_\alpha S(t)y.$$

On the other side we have

$$S_\alpha(t)\,f_{\alpha\beta}(x_\beta) = S_\alpha(t)f_{\alpha\beta}(\mathrm{pr}_\beta(y)) = S_\alpha(t)\,\mathrm{pr}_\alpha(y) = \mathrm{pr}_\alpha S(t)(y).$$

Hence it results that

$$f_{\alpha\beta}\,S_\beta(t)(x_\beta) = S_\alpha(t)f_{\alpha\beta}(x_\beta)$$

and finally we can say that $(S_\alpha)_{\alpha\in I}$ is a projective family of locally bounded semigroups. Let us denote by $T = \lim\limits_{\leftarrow} S_\alpha$. It is easy to see that $T$ is a $\mathscr{P}$-locally bounded semigroup on $\lim\limits_{\leftarrow}\mathscr{X}_\alpha$. Let $y \in Z$ and $t \geq 0$. Then

$$T(t)(y) = (S_\alpha(t)(\mathrm{pr}_\alpha(y)))_{\alpha\in I} = (\mathrm{pr}_\alpha(S(t)(y)))_{\alpha\in I} = S(t)(y).$$

This means that

$$S(t) = \left(\lim_{\leftarrow} S_\alpha(t)\right)\Big|_Z.$$

Uniqueness: Suppose that there exists another projective family of locally bounded semigroups $(T_\alpha)_{\alpha\in I}$ such that $S(t) = \left(\lim\limits_{\leftarrow} T_\alpha(t)\right)\Big|_Z$ for all $t \geq 0$.

Let $\alpha \in I$, $t \geq 0$, $x_\alpha \in \mathscr{X}_\alpha$ and $y \in Z$ such that $\mathrm{pr}_\alpha(y) = x_\alpha$. Then we have

$$\left(\lim_{\leftarrow} S_\alpha(t)\right)(y) = \left(\lim_{\leftarrow} T_\alpha(t)\right)(y).$$

Then it follows that $S_\alpha(t)(x_\alpha) = T_\alpha(t)(x_\alpha)$. Hence $S_\alpha(t) = T_\alpha(t)$ and finally $S_\alpha = T_\alpha$ for all $\alpha \in I$ and the proof is complete. ❑

**Corollary**. *If we replace in the statement of Theorem 2.3, Z by* $\lim\limits_{\leftarrow}\mathscr{X}_\alpha$ *and suppose that* $\mathrm{pr}_\alpha\left(\lim\limits_{\leftarrow}\mathscr{X}_\alpha\right) = \mathscr{X}_\alpha$ *for all* $\alpha \in I$, *then the following assertions are equivalent:*

*1. S is a* $\mathscr{P}$*-locally bounded semigroup.*

*2. There exists a unique projective family of locally bounded semigroups* $(S_\alpha)_{\alpha\in I}$ *acting on* $\mathscr{X}_\alpha$ *such that*

$$S(t) = \lim_{\leftarrow} S_\alpha(t), \textit{ for all } t \geq 0.$$

*Now we apply the previous result to locally bounded semigroups and a characterization of them is given.*

**Theorem 2.4.** *Let* $(\mathscr{X}, \mathscr{P})$ *be a Hausdorff locally convex space,* $\mathscr{P} = \{p_\alpha : \alpha \in I\}$. $(\mathscr{X}_\alpha, f_{\alpha\beta})$ *the projective family associated to* $(\mathscr{X}, \mathscr{P})$ *and* $S : [0, \infty) \to L(\mathscr{X})$ *a mapping. Then the following assertions are equivalent:*

*1. S is a* $\mathscr{P}$*-locally bounded semigroup.*

*2. There exists a unique projective family of locally bounded semigroups* $(S_\alpha)_{\alpha \in I}$ *acting on* $\mathscr{X}_\alpha$ *such that*

$$fS(t) = \left(\lim_{\leftarrow} S_\alpha(t)\right) f, \textit{ for all } t \geq 0.$$

*Proof.* Let us remark from the beginning that $f : \mathscr{X} \to \lim\limits_{\leftarrow} \mathscr{X}_\alpha$ defined by

$$f(x) = (x + J_\alpha)_{\alpha \in I}$$

fulfils the condition $\mathrm{pr}_\alpha(f(\mathscr{X})) = \mathscr{X}_\alpha$ for all $\alpha \in I$. Let now the product space $\left\{\prod\limits_{\alpha \in I} \mathscr{X}_\alpha, Q\right\}$, where $Q = \{q_\alpha : \alpha \in I\}$ and

(1) $$q_\alpha(f(x)) = \|x + J_\alpha\|_\alpha = p_\alpha(x), \ x \in \mathscr{X}, \ \alpha \in I.$$

For each $t \geq 0$ we can define the operator $T(t) : f(\mathscr{X}) \to f(\mathscr{X})$ by the formula

(2) $$T(t)f(x) = fS(t)(x), \ \ x \in \mathscr{X}.$$

From (1) and (2) it follows that $T(t) \in L(f(\mathscr{X}))$, $t \geq 0$. In addition one can deduce that: $S$ is a $\mathscr{P}$-locally bounded semigroup on $\mathscr{X}$ if and only if $T$ is a Q-locally bounded semigroup on $f(\mathscr{X})$. Now it is possible to apply Theorem 2.3 to $(\mathscr{X}_\alpha, f_{\alpha\beta})$, $Z = f(\mathscr{X})$ and $T$. Then there exists a unique projective family of locally bounded semigroups $(S_\alpha)_{\alpha \in I}$ on $\mathscr{X}_\alpha$ such that

$$T(t) = \left(\lim_{\leftarrow} S_\alpha(t)\right)\Big|_{f(\mathscr{X})}$$

or equivalently

$$T(t)f = \left(\lim_{\leftarrow} S_\alpha(t)\right) f.$$

From (2) we obtain finally the formula $fS(t) = \left(\lim\limits_{\leftarrow} S_\alpha(t)\right) f$, $t \geq 0$ and the proof is finished. ❑

**Corollary.** *If in addition in Theorem 2.4,* $(\mathscr{X}, \mathscr{P})$ *is supposed complete, then the following assertions are equivalent:*

*1. S is a* $C_0$ *and* $\mathscr{P}$*-locally bounded semigroup.*

*2. There exists a unique projective family of* $C_0$*-semigroups* $(S_\alpha)_{\alpha \in I}$ *on* $\hat{\mathscr{X}}_\alpha$ *such that* $S(t) = \lim\limits_{\leftarrow} S_\alpha(t)$, $t \geq 0$.

*Proof.* Since $\mathscr{X}$ is complet then the following conditions are fulfiled ([3], p. 70).

a) $\lim\limits_{\leftarrow} \mathscr{X}_\alpha = \lim\limits_{\leftarrow} \hat{\mathscr{X}}_\alpha$. ($\hat{\mathscr{X}}_\alpha$ is the completion of $\mathscr{X}_\alpha$).

b) $f : \mathscr{X} \to \lim\limits_{\leftarrow} \mathscr{X}_\alpha$ is an isomorphism.

In this case $T$ constructed in Theorem 2.4 is a semigroup acting on $\lim\limits_{\leftarrow} \hat{\mathscr{X}}_\alpha$, and $T(t) = \lim\limits_{\leftarrow} S_\alpha(t)$, $t \geq 0$. From (2) it follows that: $S$ is $C_0$-semigroup if and only if $T$ is a $C_0$-semigroup. On the other hand it is easy to prove that: $T$ is a $C_0$-semigroup if and only if $S_\alpha$ is a $C_0$-semigroup for all $\alpha \in I$. Consider now the extension $\hat{S}_\alpha(t) : \hat{\mathscr{X}}_\alpha \to \hat{\mathscr{X}}_\alpha$ for all $\alpha \in I$ and $t \geq 0$. It is easy to prove that $(\hat{S}_\alpha)_{\alpha \in I}$ is a projective family of $C_0$-semigroups and $\lim\limits_{\leftarrow} S_\alpha = \lim\limits_{\leftarrow} \hat{S}_\alpha$. Therefore $T(t) = \lim\limits_{\leftarrow} \hat{S}_\alpha(t)$ and by (2) it follows that

$$fS(t) = T(t)f = \left(\lim_{\leftarrow} \hat{S}_\alpha(t)\right) f,$$

that is $S(t) = \lim\limits_{\leftarrow} \hat{S}_\alpha(t)$ for all $t \geq 0$. ❑

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# ABSTRACT

# NORMAL CONTRACTIONS PRESERVING POTENTIALS

by Gh. Bucur

1. Let $(H,<,>)$ be a Hilbert space endowed with an order relation $\le$ such that $H$ becomes a vector lattice. If for any two element $x, y$ of $H$ we use the notation $x \wedge y$ (resp. $x \vee y$) to mark the greatest lower (resp. least upper) bound of the set $\{x, y\}$ then we say that $H$ is a **Dirichlet space** if for any order disjoint positive elements $x, y$ (i.e. $x \wedge y = 0$) we have $<x, y> \le 0$.

It is known (see [1], [3]) that if $H$ is a Dirichlet space then the set $P$ of all potentials i.e.

$$P := \{p \in H \mid <p,h> \ge 0,\ \forall\ h \in H_+\}$$

is an $H$-cone. Particularly (see [1]) the elements from $P$ are positive and

$$P_1, P_2 \in P \Rightarrow p_1 \wedge p_2 \in P.$$

Also, even if the vector lattice $H$ is not complete, for any subset $A$ of $P$, $A \ne \emptyset$, there exist the greatest lower bound $\wedge A$ and we have

$$\wedge A \in P,\ \ p + \wedge A = \wedge\{p+q \mid q \in A\}\ \ \forall\ p \in P$$

As for least upper bound it is known (see [3]) that any increasing and dominated family $A$ from $P$ there exist $\vee A$ and we have

$$\vee A \in P,\ p + \vee A = \vee\{p+q \mid q \in P\}\ \ \forall\ p \in P.$$

We remember also that $P$ is a closed convex cone which satisfies the Riesz decomposition property with respect to the order relation $\le$, i.e.

$$(p, q_1, q_2 \in P,\ p \le q_1 + q_2) \Rightarrow (\exists\ p_1, p_2 \in P,\ p_i \le q_i,\ p = p_1 + p_2).$$

This property is equivalent (see [5]) with the following one

$$p, q \in P \Rightarrow p - R(p-q) \in P$$

where we have denoted:

$$R(p-q) = \wedge\{t \in P \mid t \ge p - q\}$$

Returning to the families of potentials which are directed we remember the following convergence properties:

**If $(p_i)_{i\in I}$ is a lower directed family of potentials then it converges in the Hilbert space $H$ to $\wedge p_i$.**

**If $(p_i)_{i\in I}$ is an upper directed family of potentials such that there exist $M \in \mathbb{R}$ with $\|p_i\| \le M$ for all $i \in I$ then $(p_i)_{i\in I}$ converges in the Hilbert space $H$ to $\bigvee_{i\in I} p_i$.**

2. A Dirichlet space $(H,<,>,\le)$ is termed a **functional Dirichlet space** if there exist a set $X$ and a family $\mathcal{N}$ of subsets of $X$ (termed negligible) which is hereditary (i.e. $A \in \mathcal{N}$, $B \subset A \Rightarrow B \in \mathcal{N}$) and $\sigma$-closed (i.e. $A_n \in \mathcal{N} \Rightarrow \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{N}$) such that:

a) the elements of $H$ are equivalent classes of numerical functions which are finite outside of an element of $\mathcal{N}$ where the equivalent relation is given by $f \sim g \Leftrightarrow \exists\ A \in \mathcal{N}$ s.t. $f(x) = g(x)\ \forall\ x \in X \setminus A$.

b) the order relation $\le$ in $H$ coincides with the pointwise order relation in $X$ outside an element of $\mathcal{N}$.

c) for any sequence $(u_n)_n$ in $H$ which converges in the Hilbert space to $u_0$ there exist a subsequence $(u_{n_p})_{p\in\mathbb{N}}$ and an element $A \in \mathcal{N}$ such that we have

$$\lim_{p\to\infty} u_{n_p}(x) = u_0(x) \quad \forall\, x \in X \setminus A .$$

**Definition 1.** *A map* $T : \mathbb{R} \to \mathbb{R}$ *is termed a* ***real contraction*** *if*

1) $T(0) = 0$

2) $\forall\ x, y \in \mathbb{R},\ |T(x) - T(y)| \le |x - y|$.

*One of the most important real contraction is the contraction denoted by* $T_1$ *which is defined by*

$$T_1 x = x^+ \wedge 1, \quad \forall\, x \in \mathbb{R} .$$

*We say that a* ***contrqction T acts on the functional Dirichlet space*** $(H, <\ >, \le)$ *if we have*

$$f \in H \Rightarrow Tf \in H \ \ \textit{and}\ \ \|Tf\| \le \|f\| .$$

It is known (see [1], [3], [4]) that **if the contraction** $T_1$ **acts in** ***H*** **then all contractions act in** ***H*****.**

Now we consider the following question: **what are the contractions which preserve the potentials?**

The answer to this question is the following :

***Theorem*** **1. A contraction** ***T*** **on** $\mathbb{R}$ **such that** ***T*** **is concave and increasing on** $\mathbb{R}_+$**, and** $Tx = 0$ **for all** $x \le 0$**, preserves the potentials** (i.e. $Tp \in P\ \forall\ p \in P$) **in any functional Dirichlet space in which the contractions act.**

***Theorem*** **2. If** ***T*** **is a contraction on** $\mathbb{R}$ **such that** ***T*** **preserves the potentials in any functional Dirichlet space in which the contractions act then** ***T*** **is an increasing concave function on** $\mathbb{R}_+$ **and** $Tx = 0$ **for all** $x \le 0$**.**

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# A Cauchy problem involving almost periodic measures

Silvia -Otilia Corduneanu

J. Lamadrid and L. Argabright defined the almost periodic measures on a locally compact abelian group $G$. The set $ap(G)$ of all almost periodic measures is a locally convex space with respect to a topology which is called *the product topology.* In our paper we try to find the solution for the following Cauchy problem

$$\begin{cases} \dfrac{du(t)}{dt} = \nu * u(t), \quad u(0) = u_0, \\ u(0) = u_0, \end{cases}$$

where $u \in \mathcal{C}^1(\mathbb{R}, ap(G))$, $\nu$ is a bounded measure on $G$ and $u_0$ is a certain almost periodic measure.

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# OPERATOR-VALUED MOMENT PROBLEMS INVOLVING EXTENSION RESULTS

Luminita Lemnete-Ninulescu

In this note we present two complex moment problems solved by a theorem o extension of linear operators [4].The theorem is:

"*Let X be a locally convex space, let Y be an ordered complete vector lattic with strong unit and let* $X_0$ *be a vector subspace of X. Let* $A \subset X$ *be a convex subse such that the following two conditions are fulfilled:*

*(a) there exists a neighborhood V of the origin such that* $(X_0 + V) \cap A = \Phi$

*(b) A is bounded*

*Then for any equicontinuous family of linear operators* $\{f_i\}_{i\in I} \in L(X_0, Y)$ *and for any* $\bar{y} > 0$ $(\bar{y} \in Y)$ *,there exists an equicontinuous family* $\{\bar{f}_i\}_{i\in I} \in L(X,Y)$ *such that* $\bar{f}_i|_{X_0} = f_i$ *and* $\bar{f}_i|_A \geq \bar{y}, i \in I$*.Moreover, let* $u_0$ *be a strong unit in Y and let V be a convex circled neighborhood of the origin with the properties*

*(c)* $f_i(V \cap X_0) \subset [-u_0, u_0]$

*(d)* $(X_0 + V) \cap A = \Phi$*.We denote by* $p_V$ *the Minkowski functional attached to V; if we choose* $0 < \alpha \in R$ *such that* $p_V|_A \leq \alpha$ *and* $0 < \alpha_1 \in R$ *such that* $\bar{y} \leq \alpha_1 u_0$ *then the following holds:*

(e) $\bar{f}_i(x) \leq (1 + \alpha + \alpha_1) p_V(x) u_0, \forall x \in X, i \in I$."

We consider $Y=C$; on $C$ we have the order relation $z_1 \leq z_2$ iff $\operatorname{Re} z_1 \leq \operatorname{Re} z_2$ and $\operatorname{Im} z_1 \leq \operatorname{Im} z_2$. Endowed with this order relation $C$ is an order complete vector lattice with strong unit $u_0 = 1 + i$.Let $X = C_C(T)$ with T a compact set in $C, 0 \in T$ ,$X$ considered as a real vector space; X is a locally convex space. The sequence of complex numbers $\{\alpha_{n,m}\}_{n,m} \subset C$ is positive defined on T if the following implication holds: $|\sum_{n,m} \xi_{n,m} z^n \bar{z}^m| \leq 1 \forall z \in T$ implies

$-1 \leq \operatorname{Re} \sum_{n,m} \xi_{n,m} \alpha_{n,m} \leq 1$ and $-1 \leq \operatorname{Im} \sum_{n,m} \xi_{n,m} \alpha_{n,m} \leq 1$.

***Remark*** : In any situation of $T \neq \Phi$ ,there exists such a sequence.

***Problem 1*** : Let T a compact set in C with $0 \in T$ and $X = C_C(T), X_0 = sp\{z^n \bar{z}^m, n + m \geq 1\}$, A=co$\{e^{-k|z|}, z \in T\}$.We consider Y=C as before and $\{\alpha_{n,m}\}_{n,m} \subset C$ a positive defined sequence on T. In these conditions for any $\alpha \in R$ there **exists** $f \in L(X,Y)$ **such that**

(1) $f(z^n \bar{z}^m) = \alpha_{n m}, \forall n, m \in N, n+m \geq 1$

(2) $f(e^{-k|z|}) \geq \alpha_1 (1+i)$ , $k \in IN$

(3) $f(x) \leq (2+\alpha_1) \| x \| (1+i)\ \forall x \in C_C(T)$.

With a minor modification when H is a complex Hilbert vector space and $A \in A(H)$ is a selfadjoint operator on $H$ it is known from [2] that if $A_1 = \{U \in A(H). UA = AU\}$ and $A_0 = \{V \in A_1, VU = UV, \forall U \in A_1\}$ $A_0$ is a complete vector lattice with strong unit $u_0 = I$. We have the same in the bidimensional case:

Let $H$ be a complex Hilbert space and $A_0$ a part of $A(H)xA(H)$ with the following properties:

(i) If $(U_1, V_1)$ and $(U_2, V_2) \in A_0$, then $(U_1 + U_2, V_1 + V_2) \in A_0$

(ii) If $(U, V) \in A_0$ and $\lambda \in R$, then $\lambda(U, V) = (\lambda U, \lambda V) \in A_0$

(iii) If $(U, A)$ and $(V, A) \in A_0$ then $(UV, A) \in A_0$ and $UV = VU$

$(A, U)$ and $(A, V) \in A_0$, then $(A, UV) \in A_0$ and $UV = VU$

(iv) If $(U_\delta, V_\delta)_{\delta \in \Delta}$ is a generalized sequence in $A_0$ such that the component sequences $\{U_\delta\}_\delta$ and $\{V_\delta\}_\delta$ are bounded and pointwise convergent to the operators $U$ respectively $V$, then $(U, V) \in A_0$.

In these conditions, $A_0$ is a complete vector lattice and if $(I, I) \in A_0$, then $(I, I) = u_0$ is a strong unit in $A_0$.

***Concrete example*** : Let H be a complex Hilbert space and $N = U_0 + iV_0$ a normal operator on H, $U_0 = \frac{N + N^*}{2}, V_0 = \frac{N - N^*}{2i}$ such that the spectrum $\sigma(N) \subseteq B(0.1)$. We define

$A_1 = \{(U, V) \in A(H)xA(H), UU_0 = U_0 U, VV_0 = V_0 V, UV_0 = V_0 U, U_0 V = VU_0\}$ and $A_0 = \{(U, V) \in A_1$ such that $UU_1 = U_1 U, VV_1 = V_1 V, UV_1 = V_1 U, U_1 V = VU_1$ for $\forall (U_1, V_1) \in A_1\}$. In these conditions, $A_0$ is a complete vector lattice and if $(I, I) \in A_0$ then $(I, I) = u_0$ is a strong unit in $A_0$ (it satisfies conditions i-iv). Moreover, if $(U_1, V_1)$ and $(U_2, V_2)$ belongs to $A_0$, then also $(U_1 U_2 - V_1 V_2, U_1 V_2 + V_1 U_2) \in A_0$. If we identify $N \in K(H)$ with the pair $(U_0, V_0) \in A_0$ we have also $N^k N^{*m} \in A_0$ for $\forall k, m \in N$ with the same identification.

***Problem 2*** : Let $K$ a compact set in $C$ such that $0 \in K$ and $X = C_C(K)$ organized as a real vector space.

$X_0 = sp\{z^n \bar{z}^m, n+m \geq 1\} \subset X, A = co\{e^{-k|z|}, k \in N, z \in K\}, Y = A_0$ as before and $\alpha_1 > 0$. In these conditions, there exists $f \in L(X,Y)$ such that:

(1) $f(z^n \bar{z}^m) = N^n N^{*m}$ $\forall (n,m) \in N^2, n+m \geq 1$ (with the mentioned identification)

(2) $f(e^{-k|z|}) \geq \alpha_1 (I,I) = \alpha_1 u_0, \forall k \in N$

(3) $f(x) \leq (2+\alpha_1) \| x \| u_0$, where $\| x \| = \sup\{| x(t) |, t \in K\}$

*Proof*: Let $X = C_C(K)$ is locally convex , $Y = A_0$ as before is a complete vector lattice with strong unit $u_0 = (I,I)$. Because $d(X_0, A) \geq 1$, we have (d). We thus, only verify that $f(X_0 \cap B(0,1)) \subset [-u_0, u_0]$. This is :

if $| \sum_{n+m\geq 1} \xi_{n,m} z^n \bar{z}^m | \leq 1$ we must have

$-I \leq \operatorname{Re} \sum_{n+m\geq 1} \xi_{n,m} N^n N^{*m} \leq I$ and $-I \leq \operatorname{Im} \sum_{n+m\geq 1} \xi_{n,m} N^n N^{*m} \leq I$.

That is $-I \leq \dfrac{\sum_{n,m} \xi_{n,m} N^n N^{*m} + \sum_{n,m} \xi_{n,m} N^m N^{*n}}{2} \leq I$ and

$$-I \leq \frac{\sum_{n,m} \xi_{n,m} N^n N^{*m} - \sum_{n,m} \xi_{n,m} N^m N^{*n}}{2i} \leq I.$$

From the integral representation of the normal operator N with respect to the associated spectral measure $\mu$ the inequalities becomes:

$-\int_{\sigma(N)} d\mu(z) \leq \int_{\sigma(N)} \operatorname{Re}[\sum_{n+m\geq 1} \xi_{n,m} z^n \bar{z}^m] d\mu(z) \leq \int_{\sigma(N)} d\mu(z)$ and

$-\int_{\sigma(N)} d\mu(z) \leq \int_{\sigma(N)} \operatorname{Im}[\sum_{n+m\geq 1} \xi_{n,m} z^n \bar{z}^m] d\mu(z) \leq \int_{\sigma(N)} d\mu(z)$, inequalities that are true.

Q.E.D.

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# THE REPRESENTATION OF CERTAIN OPERATORS AND FUNCTIONALS AS INTEGRALS OF THE COMPOSITION OF FUNCTIONS IN ORDERED LINEAR SPACES

Gilda Moldoveanu

The following transformations of integrals are based on the results of [1], [2], [3], [4], [5], [6], [7]. The notion of $\mu$-integrable function is introduced in [1], [2] and [3]. The notions regarding order are from [2] and [4].

Let $X, Y$ be $\sigma$-regular, complete vector lattices and $R(X, Y)$ the complete vector lattice of the regular operators defined on $X$, with values in $Y$. $T$ is a set, $\mathcal{T}$ is a $\sigma$-algebra of subsets of $T$, and $\mu : \mathcal{T} \to \mathbb{R}_+$ is countably additive. $L_o(T, X, \mu)$ is the set of $\mu$-integrable functions, $G = \{f \in L_o(T, X, \mu) \mid f(t) = 0 \mu a.e.\}$, $L(T, X, \mu) = L_o(T, X, \mu) \mid G$ which is a vector lattice.

**Theorem 1:** *Let $w : T \to R(X, Y)$ be a positive function with the following properties:*

(1) $\forall x \in X$, $w_x : T \to Y$, defined by $w_x(t) = w(t)(x)$, $\forall t \in T$, is $\mu$-integrable

(2) $\exists P \in R(X, Y)$ such that $w(t) \leq P$, $\forall t \in T$.

The operator $U : L(T, X, \mu) \to Y$ is defined by

$$U(\hat{f}) = \int \langle w, f \rangle (t) d\mu(t), \quad \forall \hat{f} \in L(T, X, \mu)$$

where $< w, f > (t) = w(t)(f(t))$, $\forall\, t \in T$.

Then there exists a measure $m : \mathcal{T} \to R(X, Y)$ such that $m(A) \leq \mu(A)P$, $\forall A \in \mathcal{T}$ and for which

$$U(\hat{f}) = \int f(t)dm(t), \quad \forall \hat{f} \in L(T, X, \mu).$$

The existence of $U$ is based on theorem 3 of [7]. The existence of $m$ is proved using theorem 2 of [6].

**Theorem 2:** *Let $U : L(T, X, \mu) \to \mathbb{R}$ be defined by*

$$U(\hat{f}) = \int f(t)dm(t), \quad \forall \hat{f} \in L(T, X, \mu),$$

*where $m : \mathcal{T} \to X_o^*$ is a measure for which there exists $P \in X_o^*$ such that $m(A) \leq \mu(A)P$, $\forall A \in \mathcal{T}$.*

*Then there exists a function $w : T \to X_o^*$ with the following properties*
*(1) $\forall x \in X$, the function $w_x : T \to \mathbb{R}$, defined by $w_x(t) = w(t)(x)$, $\forall t \in T$, is $\mu$-integrable*
*(2) $0 \leq w(t) \leq P$, $\forall t \in T$*
*and $U(\hat{f}) = \int \langle w, f \rangle d\mu(t)$, $\forall \hat{f} \in L(T, X, \mu)$.*

The proof is based on prop 1/p.199 of [5] from which it follows that the space $\mathcal{L}^\infty(T, \mathcal{T}, \mu)$ has the lifting property.

The last transformation of integrals shows that the integral representation of [1] can be considered for the functionals as an integral representation of the composition of certain functions.

**Proposition 1:** *The general form of the linear positive functionals $U : L(T, X, \mu) \to \mathbb{R}$ for which there exists $P \in X_o^*$ such that*

$$|U(\hat{f})| \leq P\left(\int |f(t)| d\mu(t)\right), \quad \forall \hat{f} \in L(T, X, \mu)$$

*is given by*

$$U(\hat{f}) = \int \langle w, f \rangle d\mu(t), \quad \forall \hat{f} \in L(T, X, \mu),$$

*where $w : T \to X_0^*$ has the properties (1) and (2) from theorems 1 and 2.*

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# EXTENSION OF LINEAR OPERATORS, DISTANCED CONVEX SETS AND THE MOMENT PROBLEM

OCTAV OLTEANU

**Abstract**

One applies a general extension theorem for linear operators (theorem 5 [5], p. 969) to the classical moment problem in the spaces $C([0,b])$ and $A_b$. Our solutions fulfil some natural sandwich type conditions.

THEOREM 1. *Let* $0 < b \leq 1$, $X := C([0,b])$, $x_j(t) = t^j$, $j \in \boldsymbol{N}$, $j \geq 1$, $\{\varphi_k; k \in \boldsymbol{N}\} \subset X$, $||\varphi_k|| \leq 1$, $\varphi_k(0) = 1$, $k \in \boldsymbol{N}$. *Let* $Y$ *be an order complete vector lattice with a strong unit* $u_0$, *and let* $\{y_1, y_2, \ldots\} \subset Y$ *be such that the sequence* $\{u_0, y_1, y_2, \ldots\}$ *is positive on* $[0,b]$ ($\sum_{j=0}^{n} \lambda_j t^j \geq 0\ \forall t \in [0,b] \Rightarrow \lambda_0 u_0 + \sum_{j=1}^{n} \lambda_j y_j \geq 0$ *in* $Y$, $n \in \boldsymbol{N}$, $\lambda_j \in \boldsymbol{R}$).

Then for any $\alpha_1 \in \mathbf{R}_+$, there exists $f \in L(X,Y)$ such that

$$f(x_j) = y_j, \qquad j \in \mathbf{N},\quad j \geq 1$$
$$f(\varphi_k) \geq \alpha_1 u_0, \qquad k \in \mathbf{N}$$
$$f(x) \leq (2+\alpha_1)||x||u_0, \quad x \in X$$

In the following $X$ will be a space of analytic functions. Let $b > 0$ and $X := A_b$ the space of all functions $x$ which may be represented as an absolutely convergent series $x(z) = \sum_{j=0}^{\infty} \lambda_j z^j$, $|z| < b$, $\lambda_j \in \mathbf{R}$, $x$ being continuous in the closed disk $|z| \leq b$. For $x \in X$, we denote $||x|| := \sup\{|x(z)|; |z| \leq b\}$.

Let $x_j \in X$, $x_j(z) = z^j$, $j \in \mathbf{N}$. Let $Y = L^\infty(\Omega)$ with respect to a positive measure on $\Omega$. We denote by $u_0 \in Y$ the function $u_0(\omega) = 1\ \forall\omega \in \Omega$. For $y \in Y$, we note $||y||_\infty = \text{esssup} y$.

THEOREM 2.1. *Let $b > 1$, $\{\varphi_k; k \in \mathbf{N}\} \subset X$ such that $||\varphi_k|| \leq M$, $\varphi_k(0) = 1$, $k \in \mathbf{N}$. Let $\{y_j : j \in \mathbf{N},\ j \geq 1\} \subset Y$ be a sequence such that $||y_j||_\infty \leq b - 1$, $j \geq 1$.*

*Then for any $\tilde{y} \in Y_+$, there exists $f \in L(X, Y)$ such that*

$$f(x_j) = y_j, \qquad j \in \mathbf{N},\ j \geq 1$$
$$f(\varphi_k) \geq \tilde{y}, \qquad k \in \mathbf{N}$$
$$f(x) \leq (1 + M + ||\tilde{y}||_\infty)||x||u_0, \quad x \in X$$

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# Some Remarks upon a Paper by P. Cojuhari

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September 2001

Hardy's inequality for $l^p$ or $L^p$ spaces has been of high interest for quite a long time. The aim of this paper is to produce Hardy type inequalities in Hilbert spaces in a similar manner to Hardy's inequality for $l^2$. The starting point and also main theorem is an inequality due to P. Cojuhari [C] for a pair $T, J$ of bounded operators on a Hilbert space $\mathcal{H}$. All Hardy type inequalities that will be proved, are consequences of theorem 1.1 or its version for unbounded operators. If written for the adjoint operator this inequality yields a "conjugate" Hardy type inequality. We will denote these two inequalities by (H) and (H*). We add a technical tool to supply Cojuhari's proof using unbounded operators to produce Hardy type inequalities in $l^2$. Then we prove that the power series of the unilateral shift $V$, namely $\sum JV^n$, $\sum (V^*)^n J$ are so-convergent.

Consider now a Hilbert space $\mathcal{H}$ and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$.

**Theorem 1.1** [C] *If $T, J \in \mathcal{B}(\mathcal{H})$ and $J$ is positive, then the operator $S = J - TJT^*$ is selfadjoint and the following inequalities hold :*

*i)* $\|x\| \cdot \inf_{\|y\|=1} < Sy, y > \leq 2 \cdot \|(I-T)Jx\|$ *for all* $x \in \mathcal{H}$.

*ii)* (H) $\|Ux\| \cdot \inf_{\|y\|=1} < Sy, y > \leq 2 \cdot \|x\|$ *for all* $x \in \mathcal{H}$.

*iii)* (H*) $\|U^*x\| \cdot \inf_{\|y\|=1} < Sy, y > \leq 2 \cdot \|x\|$ *for all* $x \in \mathcal{H}$

Where $U$ is the extension of $((I-T)J)^{-1}$. By applying these inequalities to particular choices of the Hilbert space $\mathcal{H}$ and the operators $T$, $J$ we get Hardy type inequalities also denoted by (H) and (H*).

## 1. THE FINITE DIMENSIONAL APPROACH (BOUNDED OPERATORS)

Consider now the finite dimensional Hilbert space $\mathcal{H} = \mathbb{C}^p$, for $p \geq 1$, the right shift $V(\xi_1, ..., \xi_p) = (0, \xi_1, ..., \xi_{p-1})$ and the diagonal operator $J$ defined as $J(\xi_n)_{n=1,p} = (\frac{1}{n}\xi_n)_{n=1,p}$, which is clearly positive. By applying theorem 1.1 to the operators $V$ and $J^{-1}$ we get Hardy's inequality for $l^2$

$$\text{(H)} \quad \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \left|\sum_{k=1}^{n} \xi_k\right|^2\right)^{1/2} \leq 2 \left(\sum_{n=1}^{\infty} |\xi_n|^2\right)^{1/2} \quad \textit{for all} \quad (\xi_n)_{n\geq 1} \in l^2.$$

$$\text{(H*)} \quad \left(\sum_{n=1}^{\infty} \left|\sum_{k=n}^{\infty} \frac{1}{k}\xi_k\right|^2\right)^{1/2} \leq 2 \left(\sum_{n=1}^{\infty} |\xi_n|^2\right)^{1/2} \quad \textit{for all} \quad (\xi_n)_{n\geq 1} \in l^2.$$

## 2. THE INFINITE DIMENSIONAL APPROACH (UNBOUNDED OPERATORS)

**Theorem 2.1** [C] *If $T, J \in \mathcal{B}(\mathcal{H})$ and is $J$ positive with* $\ker J = \{0\}$, *then the operator $S = J^{-1} - TJ^{-1}T^*$ is symmetric and the following inequalities hold :*

*i)* $\|x\| \cdot \inf_{\|y\|=1} < Sy, y > \leq 2 \cdot \left\|(I-T)J^{-1}x\right\|$ *for all* $x, y \in \mathcal{D}_S$.

*ii)* (H) $\|Ux\| \cdot \inf_{\|y\|=1} < Sy, y > \leq 2 \cdot \|x\|$ *for all* $x \in \mathcal{H}$.

*iii)* (H*) $\|U^*x\| \cdot \inf_{\|y\|=1} < Sy, y > \leq 2 \cdot \|x\|$ *for all* $x \in \mathcal{H}$.

Where $U$ is the extension of $\left((I-T)J^{-1}\right)^{-1}$. Consider now the Hilbert space $\mathcal{H} = l^2$, the unilateral shift $V$ and the diagonal operator $J$ defined as $J(\xi_n)_{n\geq 1} = (\frac{1}{n}\xi_n)_{n\geq 1}$ for $(\xi_n)_{n\geq 1} \in l^2$.

**Proposition 2.4** *The linear operator $T : l^2 \to l^2$ defined as $T\xi \stackrel{def}{=} \sum_{n=0}^{\infty} JV^n\xi$ is bounded on $l^2$, $\|T\| \leq 2$ and $T^*\xi = \sum_{n=0}^{\infty}(V^*)^n J\xi$. We also have $T\xi = J(I-V)^{-1}\xi$ for $\xi \in \mathcal{R}_{(I-V)}$*, and $T^*\xi = (I-V^*)^{-1}J\xi$ *for $\xi \in l^0$*

**Corollary 2.5**[C] *Let V and J defined as before. Then we get Hardy's inequality for a dense subset:*

(H) $\left(\sum_{n=1}^{\infty} \frac{1}{n^2} \left|\sum_{k=1}^{n} \xi_k\right|^2\right)^{1/2} \leq 2\left(\sum_{n=1}^{\infty} |\xi_n|^2\right)^{1/2}$ *for all* $(\xi_n)_{n\geq 1} \in \mathcal{R}_{(I-V)}$

**Proposition 2.6** *In the above stated context we have:*

*i)* $\left(so-\lim_{n\to\infty} \sum_{k=0}^{n} JV^k\right)\xi = \sum_{n=0}^{\infty} JV^n\xi$ for all $\xi \in \mathcal{R}_{(I-V)} \subset l^2$

*ii)* $\left(so-\lim_{n\to\infty} \sum_{k=0}^{n} (V^*)^k J\right)\xi = \sum_{n=0}^{\infty} (V^*)^n J\xi$ for all $\xi \in l^1 \subset l^2$

## 3. Improvement tentatives

**Proposition 3.1** *Let* $(\alpha_n)_{n\geq 1}$ *be a strictly decreasing sequence with* $\alpha_n > 0$ *for* $n \geq 1$ *and* $\inf\{\frac{1}{\alpha_1}, (\frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}}), n \geq 2\} = \beta$ , *with* $0 < \beta < \infty$ . *Then:*

$$\text{(H)} \quad \beta\left(\sum_{n=1}^{\infty} \alpha_n^2 \left|\sum_{k=1}^{n} \xi_k\right|^2\right)^{1/2} \leq 2\left(\sum_{n=1}^{\infty} |\xi_n|^2\right)^{1/2} \quad \textit{for all} \quad (\xi_n)_{n\geq 1} \in l^2$$

$$\text{(H*)} \quad \beta\left(\sum_{n=1}^{\infty} \left|\sum_{k=n}^{\infty} \alpha_k \xi_k\right|^2\right)^{1/2} \leq 2\left(\sum_{n=1}^{\infty} |\xi_n|^2\right)^{1/2} \quad \textit{for all} \quad (\xi_n)_{n\geq 1} \in l^2$$

**Proposition 3.2** *The best form for the Hardy type inequality (H) or (H*) (in proposition 3.1) corresponds to the choice* $\alpha_n = \frac{1}{n}$ , $n \geq 1$ *i.e.* $J(\xi_n)_{n\geq 1} = (\frac{1}{n}\xi_n)_{n\geq 1}$.

## 4. Hardy type inequalities in $L^2(\mathbb{R}_+)$

Consider now the Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+)$, $V$ a shift type operator defined as

$V\varphi(x) = \begin{cases} 0 & , \text{for } x \in [0,1) \\ \varphi(x-1) & , \text{for } x \in [1,\infty) \end{cases}$ , for $\varphi \in L^2(\mathbb{R}_+)$.

and the operator $J$ defined as $J\varphi(x) = \frac{1}{x+1}\varphi(x)$ for all $\varphi \in L^2(\mathbb{R}_+)$.

**Theorem 4.1** [C] *In the above stated conditions, the following Hardy type inequalities hold :*

$$\text{(H)} \quad \left(\sum_{n=0}^{\infty} \int_{n}^{n+1} \left|\sum_{k=0}^{n} \frac{\varphi(x-k)}{x+1}\right|^2 dx\right)^{1/2} \leq 2\left(\int_{0}^{\infty} |\varphi(x)|^2\, dx\right)^{1/2} \quad \textit{for all} \quad \varphi \in \mathcal{R}_{(I-V)}$$

$$\text{(H*)} \quad \left(\int_{0}^{\infty} \left|\sum_{n=0}^{\infty} \frac{\varphi(x+n)}{x+n+1}\right|^2 dx\right)^{1/2} \leq 2\left(\int_{0}^{\infty} |\varphi(x)|^2\, dx\right)^{1/2} \quad \textit{for all } \varphi\ ; \quad J\varphi \in \mathcal{R}_{(I-V^*)}$$

To complete the proof for inequality (H) we need to build a context similar to the "finite dimensional" one in section 1. Just as in section 2, we get a similar result to prop. 2.6.

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# PARETO EFFICIENCY, CHOQUET BOUNDARIES AND OPERATORS IN HAUSDORFF LOCALLY CONVEX SPACES


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**Abstract**

This research paper is devoted to develop the study of the properties for Pareto type efficient points sets in separated locally convex spaces, being based on the first result established by us on the coincidence of Pareto type efficient points sets and the Choquet boundaries and its natural corresponding extension for the approximative efficient points sets in Hausdorff locally convex spaces, both of these results representing also an important connection between two great fields of Mathematics: Vector Optimization and the Potential Theory . Thus,if A is a non-empty arbitrary choosen subset of $E$ and $a_0 \in A$, then


**Definition 1.** *We say that $a_0$ is a K-efficient point of A (a Pareto type minimum for A with respect to K), in notation, $a_0 \in eff(A,K)$ (or $a_0 \in MIN_K(A)$) if it satisfies one of the following equivalent conditions:*

*(i)* $A \cap (a_0 - K) \subseteq a_0 + K$; *(ii)* $K \cap (a_0 - A) \subseteq -K$;

*(iii)* $(A+K) \cap (a_0 - K) \subseteq a_0 + K$; *(iv)* $K \cap (a_0 - A - K) \subseteq -K$.

Whenever $K$ is pointed, that is, $K \cap (-K) = \{0\}$, then $a_0 \in eff(A,K)$ means that $a_0$ fulfiles one of the next equivalent relations:

a) $A \cap (a_0 - K) = \{a_0\}$; *(b)* $K \cap (a_0 - A) = \{0\}$;

*c)* $A \cap (a_0 - K \setminus \{0\}) = \varnothing$; *(d)* $(K \setminus \{0\}) \cap (a_0 - A) = \varnothing$.

In a similar manner one defines the Pareto type maximum elements of $A$. In fact, $a_0' \in a$ is a Pareto type maximum point for $A$ with respect to $K$, in notation, $a_0' \in MAX_K(A)$ if it is a Pareto minimum point of $A$ with respect to $-K$, that is, $a_0' \in eff(A,-K)$, i.e. $a_0' \in MIN_K(A)$.

The immediate connection with the fixed points for multifunctions is obviously contained in

**Remark 1.** $a_0 \in eff(A,K)$ is and only if it is a fixed point for at least one of the following multifunctions:

$$F_1 : A \to A,\ F_1(t) = \{\alpha \in A : A \cap (\alpha - K) \subseteq t + K\},$$

$$F_2 : A \to A,\ F_2(t) = \{\alpha \in A : A \cap (t - K) \subseteq \alpha + K\},$$

$$F_3 : A \to A,\ F_3(t) = \{\alpha \in A : (A+K) \cap (\alpha - K) \subseteq t + K\},$$

$$F_4 : A \to A,\ F_4(t) = \{\alpha \in A : (A+K) \cap (t - K) \subseteq \alpha + K\},$$

that is, $a_0 \in F_i(a_0)$ for same $i = \overline{1,4}$.

**Remark 2.** It is known that, if $A \subseteq E$ is an arbitrary non-empty set, then a point-to-set mapping $\Gamma : A \to 2^A$ is called a *generalized dynamical system* when $\Gamma(x) \neq \varnothing$

for every $x \in A$. A point $a_0 \in A$ is said to be a *critical (sometimes equilibrium point* for $\Gamma$) if $\Gamma(a_0) = \{a_0\}$. It is easy to see that that whenever $K$ is a pointed, convex cone in $E$, then $a_0 \in eff(A,K)$ if and only if $a_0$ is a critical point for the generalized dynamical system $\Gamma$ defined by $\Gamma(a) = A \cap (a-K), a \in A$. Thus, one can says that $eff(A,K)$ describes a state of equilibrium for $\Gamma$ and the ideal equilibria are contained in this set. If $X$ is any non-empty, compact subset of $E$ and $K$ is an arbitrary closed, convex pointed cone, then

**Theorem 1.** *$eff(X,K)$ coincides with the Choquet boundary of $X$ with respect to the convex cone of all real continuous functions which are increasing with respect to order relation $\leq_K$. Consequently, the set $eff(X,K)$ endowed with the trace topology $\tau_X$ induced on $X$ by $\tau$ is a Baire space. Moreover, if $X$ is metrizable, then $eff(X,K)$ is a $G_\delta$ - set in $(X,\tau_X)$.*

**Corollary 1.1**

*(i)* $eff(X,K) = \{x \in X : f(x) = \sup\{f(x') : x' \in X \cap (x-K)\}$ *for all* $f \in C(X)\}$.

*(ii)* $eff(X,K)$ *and* $eff(X,K) \cap \{x \in X : s(x) \leq 0\}$ $(s \in S)$ *are compact sets with respect to Choquet's topology;*

*(iii)* $eff(X,K)$ *is a compact subset of X.*

**Remark 3.** There exists more general conditions than compactness imposed upon a non-empty set $A$ in a separated locally convex space ordered by a convex cone $K$ ensuring that $eff(X,K) \neq \varnothing$. Perhaps our coincidence result suggests a natural extension of the Choquet boundary at least in these cases.

**Definition 2.** *If a non-empty subset of E, then $a_0 \in A$ will be called an minimal element ($\varepsilon$ – efficient point, Pareto $\varepsilon$ – efficient point, $\varepsilon$ – near to minimum point) of $A$ with respect to $K$ if there exists no $a \in A$ such that $a_0 - a - \varepsilon \in K$, that is, $(a_0 - \varepsilon - K) \cap A = \varnothing$.*

The $\varepsilon$ – efficient points set of A with respect to K will be denoted by $\varepsilon - eff(A,K)$,

**Remark 4.** It is clear that the concept of the $\varepsilon$ – efficient point does not include the notion of efficient point, $eff(A,K) \subseteq \varepsilon - eff(A,K), \forall \varepsilon \in K \setminus \{0\}$ and $eff(A,K) = \bigcap_{\varepsilon \in K \setminus \{0\}} [\varepsilon - eff(A,K)]$.

**Definition 3.** *A real function $f: E \to R$ is called $\varepsilon + K$ – increasing if $f(x_1) \geq f(x_2)$ whenever $x_1, x_2 \in E$ and $x_1 \in x_2 + \varepsilon + K$.*

**Theorem 3.** *If X is any non-empty and compact subset of E, then the set $\varepsilon - eff(X,K)$ coincides with the Choquet boundary of X with respect to the convex cone and all $\varepsilon + K$ – increasing real continuous functions on X. Consequently, the set $\varepsilon - eff(x,K)$ endowed with the trace topology is a Baire space and if $(X,\tau_x)$ is metrizable, then $\varepsilon - eff(x,K)$ is a $G_\delta$ – subset of X.*

The paper includes also some connections with Lototsky-Schnabl operators, Altomare projections and relevant references.

# BERNSTEIN OPERATORS OF SECOND KIND

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Bernstein operators of second kind have been introduced and studied in [1]. Other properties of them have been established in [2].

In this paper we study the associated blending system, preservation properties and Voronovskaya type properties.

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# ON SOME APPROXIMATION PROCESSES IN LOCALLY CONVEX CONES

**Ligia – Adriana Sporiş**

- The aim of this paper is to present a Korovkin system for a cone of weighted continuous set – valued functions.

## § 1. Preliminaries

- Let $(G,V)$ be a separated locally convex cone such that $G$ is a linear space. We shall consider:

(1) $CConv(G) = \{ A \subset G \mid \emptyset \neq A \in CConv(G), \downarrow, \textit{compact in the upper topology on } G \}$,

which becomes a locally convex cone, as a subcone of the full locally convex cone $(\overline{DConv(G)}, \overline{V})$, where $\overline{V} = \{ \overline{v} \mid \overline{v} = \{v\}, v \in V \}$.

It's not difficult to verify that $(CConv(G), \overline{V})$ is a $M$-uniformly up-directed cone, $\vee$-semilattice and all its elements are bounded.

Recall that there's a natural embedding $j : G^* \to (CConv(G))^*$, $j(\mu) = \overline{\mu}$, where $\overline{\mu}(A) = sup\{ \mu(a) \mid a \in A \}$, $A \in CConv(G)$.

Let (2) $M = \{ \overline{\mu} \in (CConv(G))^* \mid \mu \in G^* \}$. Then, $M$ has the following properties:

1. $(\forall)\ v \in V,\ M \cap \overline{v}^0,\ \overline{v}^* - \textit{compact;}$
2. $(\forall)\ A, B \in CConv(G),\ (\forall)\ v \in V$ *pentru care* $(\exists)\ \rho > 1$ *a.i.* $A \leq B + \rho\overline{v}$, $(\exists)\ \overline{\mu} \in M \cap \overline{v}^0$ *a.i.* $\overline{\mu}(A) > \overline{\mu}(B) + 1$.
3. $\overline{\mu}(\overline{co}\{ a_1, \ldots, a_n \}) = \bigvee_{i=0}^{n} \mu(a_i)$, $a_i \in G$, $i = \overline{1,n}$.

- Let $X$ be a locally compact Hausdorff space and $w$, a weight on $X$.

Now, we shall consider:

**(3)** $C^w(X; CConv(G)) = \{ f \in C_s(X; CConv(G)) \mid (\forall) v \in V, (\exists) J \subset X, \textit{ compact such that } f \leq \overline{v}_w \textit{ and } 0 \leq f + \overline{v}_w \textit{ on } X \setminus Y \}$

endowed with abstract neighborhood system $\overline{V}_w = \left\{ \overline{v}_w \mid \overline{v}_w = \left\{ \frac{v}{w} \right\}, v \in V \right\}$ and

**(4)** $M_X^w = \left\{ \overline{\mu}_x \mid \overline{\mu}_x \in (C^w(X; CConv(G)))^*,\ \mu \in M,\ x \in X \right\}$.

Then, it can easily be proved that (3) and (4) inherit the some properties as (1) and (2).

## § 2. A Korovkin system for $C^w(X; CConv(G))$

- Firstly, we consider

**(5)** $F^w(X; CConv(G)) = \{ f \in C^w(X; CConv(G)) \mid (\exists) \varphi_i \in C^w(X; G) \textit{ of finite rank, } i = \overline{1,n}$ *such that,* $(\forall) x \in X,\ f(x) = \overline{co}(\varphi_1(x), \ldots, \varphi_n(x)) \} \left( \stackrel{not}{=} F^w \right)$.

- Prop. 1: $F^{w}$ is a sup-stable subcone of $\left(C^{w}(X;CConv(G)),\overline{V}_{w}\right)$.
- Cor. 2: $F^{w}$ is an $M$-uniformly up-directed cone and $\vee$-semilattice.
- Prop. 3: If $f \in C^{w}(X;CConv(G))$, $(\forall)v \in V$, $\mu \in G^{*}$, $x \in X$, $(\exists)g \in F^{w} \cap \overline{v}_{w}$ such

that $\overline{\mu}_{x}(f)=\overline{\mu}_{x}(g)$.
- Theorem 4: $F^{w}$ is a lower-Korovkin system for $C^{w}(X;CConv(G))$.

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# On some inequalities for operators

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Let $(X, \|\cdot\|)$ be a normed Abelian group and let $\|\cdot\|^*$ be an other norm. For all $x \in X$, the sequence $\{E_n\}_n$ is defined as follows:

$$E_n(x) := \inf\{\|x-y\| : y \in X, \|y\|^* < n\}, n = 1, 2, ...$$

If $X = L(E)$, the space of all linear and bounded operators $T : E \to E$, where $E$ is a normed space, the norm $\|\cdot\|^*$ is $\|T\|^* = rankT$ and

$$E_n(x) = a_n(T) = \inf\{\|T - A\| : rankA < n\}.$$

If $B : X \times X \to X$ is a bilinear and bounded operator, it is well known that the following inequalities hold:

$$(1) \sum_{n=1}^{k} \frac{E_n(B(x,y))}{n} \leq 6 \cdot \sum_{n=1}^{k} \frac{E_n(x) + E_n(x)}{n}, k = 1, 2, ...$$

If $\|B(x,y)\| \leq \|x\| \cdot \|y\|$ then $\|x_0\| < n$, $\|y_0\| < n$ imply that

$$\|B(x_0, y_0)\| \leq n^2.$$

(Here, without losing the generality, we shall suppose that $\|x\| < 1, \|y\| < 1$.)

For the special case when $X = L(E)$, the operator $B$ may be the tensor product operator $T_1 \otimes T_2 \in L(E \otimes_\alpha E)$, where $\alpha$ is a tensor norm, [1], [2].

In this case, the inequality (1) is of the form:

$$(2) \sum_{n=1}^{k} \frac{a_n(T_1 \otimes T_2)}{n} \leq 6 \cdot \sum_{n=1}^{k} \frac{a_n(T_1) + a_n(T_2)}{n}, k = 1, 2, ...$$

If there are considered $r$ operators, $r \geq 3$, by reiteration, the inequality (2) is(in this case)

$$(3)\ \sum_{n=1}^{k} \frac{a_n\left(\overset{r}{\underset{i=1}{\otimes}} T_i\right)}{n} \leq 6^{r-1} \cdot \sum_{n=1}^{k} \frac{\sum_{i=1}^{r} a_n(T_i)}{n}, k = 1, 2, ...$$

Unfortunately the constant $6^{r-1}$ is far to be optimal. But, by a direct computation, we can obtaine the inequality:

$$(4)\ \sum_{n=1}^{k} \frac{a_n\left(\overset{r}{\underset{i=1}{\otimes}} T_i\right)}{n} \leq 2^{r-1}(2^r - 1) \cdot \sum_{n=1}^{k} \frac{\sum_{i=1}^{r} a_n(T_i)}{n}, k = 1, 2, ...$$

**Remarks 1.** We shall recall that, here, $\|T_i\| < 1$, $i = 1, 2, ...r$. If this condition is not fullfiled, the constant $2^{r-1}(2^r - 1)$ whould be replaced by $2^{r-1}(2^r - 1) \cdot c$, where $c = \left(\max_i \{\|T_i\|\}\right)^{r-1}$.

**2.** I think that the factor $(2^r - 1)$ is not the best.

**Conjecture** The factor $(2^r - 1)$ may be replaced by $r$.

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# SUR CERTAINES THÉORÈMES DU POINT FIXE COMMUN

Florica Voicu

**Definition 1.** *Soient X un espace lineaire completement réticulé,* $d: X \times X \to X_+$ *une métrique vectorielle et A, B* $\in \mathscr{P}_{f,b}(X)$. *Soit* $\varepsilon > 0$ *et* $\bar{u}$ *est élément unité. On note par:*

$$N(\varepsilon, A) = \{x \in X,\ d(x, A) \leq \varepsilon \bar{u}\}$$

$$E(A,B) = \{\varepsilon \bar{u};\ A \subset N(\varepsilon, B),\ B \subset N(\varepsilon, A)\}$$

$$H: \mathscr{P}_{f,b}(X) \times \mathscr{P}_{f,b}(X) \to X_+$$

$$H(A,B) = \inf E(A,B).$$

**Definition 2.** *Soient X un espace linéaire complétement réticulé et* $T: X \to \mathscr{P}(X)$. *On dit que T est une* ***application multiforme*** *(bref: m-application) definie sur X à valeurs en soi même et on note par* $T: X \multimap X$.

**Definition 3.** *Soit* $f: X \to X$ *et* $T: X \to \mathscr{P}_{f,b}(X)$. *On dit que le point* $x \in X$ *est* ***un point de coïncidence pour f et T si*** $fx \in Tx$. *Si xhaque* $x \in X$ *est un* ***point de coïncidence pour f et T alors f s'appele la sélection de T.***

*Un point* $x \in X$ *s'appele* ***point fixe pour T si*** $x \in Tx$.

*On note* $C_T = \{f: X \to X \mid TX \subset fX \text{ et } fTx = Tfx,\ \forall\ x \in X\}$

*T et f s'appelent* ***applications commutatives*** *si quel que soit* $x \in X$, $f(Tx) = fTx = T(fx)$.

**Lemme 1. (Dube (1975)).** *Soit S, T* : $\to \mathscr{P}_{f,b}(X)$ *et* $x_0, x_1 \in X$. *Alors, pour tout point* $y \in T(x_1)$ *on a:*

(1) $\quad d(y, Sx_0) \leq H(Tx_1, Sx_0)$

Le résultat suivant elargit le théoreme de Banach aux applications multiformes satisfaisant les conditions de type contractive.

**Théorème 2.** *Soient X espace linéaire completement réticulé avec l'unité forte, S, T:* $X \to \mathscr{P}_{f,b}(X)$, $f \in C_S \cap C_T$, $U: X \to X_+$ *(o)-continue, inversable et isotone. On suppose que pour tout x, y* $\in X$ *on a:*

(2)
$$\begin{aligned} H(Sx,Ty) \leq\ & \alpha\, d(fx,fy) + \beta\{d(fx,Sx) + d(fy,Ty)\} \\ & + \gamma\{d(fx,Ty) + d(fy,Sx)\} + \\ & + \delta U^{-1}(\bar{u} + d(fx,fy))\, d(fx,Sx)\, d(fy,Ty) \end{aligned}$$

*où* $\alpha,\beta,\gamma,\delta \geq 0$ *et* $0 < \dfrac{\alpha+\beta+\gamma}{1-\beta-\gamma-\delta} < 1$.

*Alors, il existe un point de coïncidence commun pour f et T et pour f et S.*

**Corollaire 3.** *Soient S, T* : $X \to \mathscr{P}_{f,b}(X)$ *des applications multiforme (o)-continues et* $f: X \to X$, $f \in C_S \cap C_T$ *(o)-continue satisfaisant la condition (2).*

Si $f(z) \in Sz \cap Tz$ il implique:

*(3)* $\lim_{n\to\infty} f^n z = y^*$

alors, y* et un point fixe commun pour S, T et f.

Les mathematiciens Meir et Keeler (1969) ont établi un remarcable théorème du point fixe pour un application $T:X\to X$, $(X,d)$, satisfaisant la suivante condition:

(4) $\forall\,(\exists)\delta>0$ t.q. $\varepsilon\leq d(x,y)<\varepsilon+\delta$ il implique $d(Tx, Ty) < \varepsilon$

Un autre resultat est donné par Park et Bae (1981) pour f, $T:X\to X$, $fT=Tf$ satisfaisant:

(5) $\forall\,(\exists)\delta>0$ t.q. $\varepsilon\leq d(fx,fy)<\varepsilon+\delta$

il implique $d(Tx, Ty) < \varepsilon$ et $Tx=Ty$ quand $fx=fy$.

La technique de Meir-Keeler a été elargi aux applications contractives multiformes dans les espaces métriques par J. Siegel, L.S. Dube, K. Iseki, S.B. Nadler, S. Reich, B.K. Ray, I. Rus, T. Hu, I. Beg et A. Azam, etc.

Par la suite on donné un théorème du point de type Meir-Keeler pour les applications compatibles.

***Definition** 4. Soient X espace linéaire complétement réticulé,* $f:X\to X$, $T: X\to \mathscr{P}_{f,b}(X)$. On dit que f et T sont *compatibles* si la suite $\{x_n\}_{n\in N}\subset X$ satisfait la *condition suivante*:

(6) $\lim_{n\to\infty} fx_n \in \lim_{n\to\infty} Tx_n \Rightarrow \lim_{n\to\infty} H(fTx_n, Tfx_n)=0$.

**Lemme 3 (Hu (1980)).** *Soient X espace linéaire complétement réticulé,*

*$\{A_n\}_n\subset \mathscr{P}_{f,b}(X)$ et $H(A_n, A)\xrightarrow{0} O$ pour $A\in \mathscr{P}_{f,b}(X)$.*

*Si $x_n\in A_n$ et $d(x_n,x)\xrightarrow{0} O$* alors $x\in A$.

***Théoreme** 4. Soient X espace linéaire completement réticulé avec une norme monotone, $T:x\to \mathscr{P}_{f,b}(X)$ et $f:X\to X$ compatibles et satisfaisant les suivantes conditions:*

$$(7)\quad \begin{cases} \forall\,(\exists)\,\varepsilon>0\,(\exists)\delta>0 \text{ t.q. } \varepsilon\leq \|d(fx,fy)\| \leq \varepsilon+\delta \\ \text{il implique } \|d(v,w)\|<\varepsilon, v\in Tx,\ w\in Ty \text{ et } Tx=ty \text{ quand } fx=fy \end{cases}$$

*Si f est (o)-contimue, alors f et T ont un point fixe commun.*

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*R. Cristescu*

- On some linear operators and on some vector integrals (Symposium, June 25, 1999).
- Vector integrals in vector normed spaces (Symposium, June 29, 2000).

*N. Dăneţ*

- Spaces of regular operators with the Riesz property.

*R.M.Dăneţ*

- Lattice-vector operators and operators of lattice-vector type (Symposium, June 25 1999)
- On the extension of some positive operators.

*W.Farkaş*

- Sobolev spaces associated to certain negative definite continuous functions.

*G.Grigore*

- Functional equations in topological ordered linear spaces (Symposium, June 25, 1999)

*G.Grigore* and *D.Stănică*

- An algorithm for the pseudoinverse (Symposium, June 29, 2000).

*P.Iliaş*

- Eigenvalues for $p$-Laplacian in the von Neumann problem.
- Tensor products of locally convex spaces.

- Regular operators in the Banach lattices.

- Nonlinear equations with $p$-Laplacian.

*G.Moldoveanu*

- On the representation of some linear operators.

- Integral representation of some positive operators.

*C.Niculescu*

- New considerations on the Newton inequality.

- Hardy - Littlewood - Landau inequalities.

- The Newton's inequalities (Symposium, June 29, 2000).

- Convexity associated to averages.

*L.Pavel*

- Hypergroups with the property $(T)$ of Kazhdan.

*G.Păltineanu*

- Frontal ideals and antisymmetric ideals in locally convex lattices (Symposium, June 25, 1999).

- Generalization of the theorem of Alain-Bernard concerning the frontal set with respect to a closed vector subspace (Symposium, June 29, 2000).

*I.Polyrakis* (Athens. Greece)

- Lattice subspaces.

*N.Popa*

- Dyadic Hardy spaces (Symposium. June 25. 1999).
- Matriceal harmonic analysis (Symposium, June 29, 2000).
- Some topics in matriceal analysis using vector-valued functions.

*G.Popescu*

- Order relations in $C^*$-algebras and in operator algebras.
- Positif operators in $C^*$-algebras.
- An inequality of type Schwartz in non-commutative $C^*$-algebras.

*L.Sporiş*

- On Korovkin cones in locally convex lattices.
- Quantitative aspects of the convergence of the Korovkin approximation sequences in locally convex cones.

*D.Stănică*

- Pseudoinverse of a linear applications.
- Cubature formulas on the $n$-dimensional simplex.

- *Florica Voicu* (Bucharest) - Points fixes communes pour applications multiformes.

- *Gheorghe Popescu* (Craiova) - Power series of the unilateral shift and Hardy inequalities.

- *Ligia Adriana Sporiş* (Constanţa) - On some approximations processes in locally convex cones.

11:30 - 12:00 Coffee break

12:00 - 13:30 Communications

Chairman : *Paolo Terenzi*

- *Ileana Bucur* (Bucharest) - Derivability of the set functions.

- *Vasile Postolică* (Bacău) - Pareto efficiency. Choquet boundaries and operators in Hausdorff locally convex spaces.

- *Luminiţa Lemnete - Ninulescu* (Bucharest) - Operators valued moments problems involving extension results.

- *Ion Chiţescu* (Bucharest) - Absolute continuity and Radon-Nykodim representation into functional frameworks.

Friday. September 28

Social Programme

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ROMANIAN ACADEMY

MATHEMATICAL SECTION

TECHNICAL UNIVERSITY OF CIVIL ENGINEERING

OF BUCHAREST

DEPARTMENT OF MATHEMATICS

18th Colloquium

# TOPOLOGICAL ORDERED LINEAR SPACES

Bucharest, September 26-28, 2001

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The place of the Colloquium : Bucharest, Club of Scientists 9, Lahovari Square. September 26-28, 2001

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EDEN BUSINESS GROUP, Bucureşti, Calea Victoriei nr.135, Sector 1.

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## Colloquium schedule

Wednesday, September 26

9:30 - 11:30 Communications

Chairman : *Romulus Cristescu*

- *Romulus Cristescu* (Bucharest) - On the extensions of some positive functionals and on extensible regular operators.
- *Gavril Păltineanu* and *Dan Tudor Vuza* (Bucharest) - Some approximation results for locally convex lattices.
- *Nicolae Popa* (Bucharest) - Some matrix Banach spaces.
- *Mihai Voicu* (Bucharest) - Locally bounded semigroups.
- *Rodica - Mihaela Dăneţ* (Bucharest) - A Hahn-Banach theorem for the extension of Riesz homomorphisms.
- *Nicolae Dăneţ* (Bucharest) - Some remarks on lattice subspaces.
- *Constantin Niculescu* (Craiova) - Hermite - Hadamard inequality for functions of a vector variable.

11:30 - 12:00 Coffee break

12:00 - 14:00 Communications

Chairman : *Nicolae Popa*

- *Ioannis Polyrakis* (Athens, Greece) - Geometry of cones and theory of Banach spaces.
- *Gilda Moldoveanu* (Bucharest) - The representation of certain operators and functionals as integrals of the functions in ordered linear spaces.
- *Marinică Gavrilă* (Bucharest) - Fonctions vectorielles derivable (o)-convexes.
- *Ioan Raşa* (Cluj) and *Tiberiu Vladislav* (Bucharest) - Bernstein operators of second kind.
- *Liliana Pavel* (Bucharest) - Induced representation of hypergroups and positive definite measures.
- *Octav Olteanu* (Bucharest) - Extension of linear operators distanced convex sets and the moment problem.

Thursday, September 27

9:30 - 11:30 Communications

Chairman : *Ioannis Polyrakis*

- *Paolo Terenzi* (Milano, Italy) - The basis of the general separable Banach space.
- *Gheorghe Bucur* (Bucharest) - Transformations acting in Dirichlet spaces.
- *Silvia Corduneanu* (Iaşi) - A Cauchy problem involving almost periodic measures.
- *Nicolae Ţiţa* (Braşov) - On some inequalities for operators.

*V. Timofte*

- An unicity theorem for a mechanical model.

*M. Voicu*

- Resolvents on locally convex spaces (Symposium, June 25, 1999).
- Projective limits of linear operators (Symposium, June 29, 2000).

*D. T. Vuza*

- Strongly modular and strongly latticial classes of regular operators (Symposium, June 25, 1999).

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