Eytan Agmon

Diatonicism and Farey Series

Consider the following series of fractions:

$$\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}.$$

In this series, known in number theory as the *Farey series of order 5* ($\mathbf{F}_5$), we have in ascending order every irreducible fraction between 0 and 1 (inclusive) of which the denominator does not exceed 5. $\mathbf{F}_5$, of course, is a special case. For any integer n>0 there exists a Farey series $\mathbf{F}_n$, namely, the ascending series of irreducible fractions between 0 and 1 with a denominator not exceeding n. Farey series have a number of interesting properties. A particularly well-known property is the following: if $\frac{c}{d}$ immediately follows $\frac{a}{b}$ in some Farey series $\mathbf{F}_n$, then the relation ad - bc =-1 holds (for example, in $\mathbf{F}_5$ we have $0 \cdot 5 - 1 \cdot 1 = -1$, $1 \cdot 4 - 5 \cdot 1 = -1$, $1 \cdot 3 - 4 \cdot 1 = -1$, $1 \cdot 5 - 3 \cdot 2 = -1$, $2 \cdot 2 - 5 \cdot 1 = -1$, and so forth); conversely, if a, b, c, and d are integers satisfying the relation ad-bc=-1, then $\frac{c}{d}$ immediately follows $\frac{a}{b}$ in a Farey series whose order is the larger of the two denominators.

One area of musical research where Farey series may be encountered is diatonic intonation. A problem that often arises in the theory of diatonic intonation is approximating rationally the irrational pure intervals, for example, the pure "fifth" $\log_2 1.5$. Approximating irrationals by rationals is a classic problem in number theory. As is well known, there is no limit to how closely one may approximate rationally a given irrational μ. In particular, if μ is between 0 and 1, one can write an infinite series of fractions beginning with 1/2 where each successive fraction is a more accurate approximation of μ. Since the denominator can only increase from one such fraction to the next, any two successive fractions in the series, say $\frac{a}{b}$ and $\frac{c}{d}$, are by definition adjacent terms in $\mathbf{F}_d$. The following is an example of such a series, where $\mu=\log_2 1.5$ (the pure "fifth"); the reader may easily verify that the relation ad-bc=±1 is satisfied for any two adjacent terms:

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{7}{12}, \frac{17}{29}, \frac{24}{41}, \ldots$$

The appearance of Farey series in connection with diatonic intonation is far from surprising, given the nature of the problem involved (i.e., approximating irrationals by rationals). In the present article, however, quite a different connection between diatonicism and Farey series shall be considered. As this connection concerns a level of diatonic reality other than the familiar level of log frequency ratios, some preliminary discussion is necessary.

It is readily demonstrated that log frequency ratios do not suffice in order to capture the intuitive sense of the notion "diatonic interval" (e.g., "minor third,"

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"perfect fifth," etc.). A dramatic example of this insufficiency is the interval of the octave. It is plainly evident that the number log2 fails to capture the crucial sense of "equivalence" inherent in the octave, as well as the intuitively significant sense by which the term "octave" is derived from the number 8. It follows that in order to describe diatonic intervals adequately it is necessary to invoke a level of diatonic reality other than the familiar level of log frequency ratios; the term "cognitive" shall be used to refer to this additional level (Agmon, 1990).

In Agmon (1989) it has been suggested that the cognitive level of diatonic reality consists of integer-pairs (s,t) mod (12,7), s=0, 1,..., 11, and t=0, 1,..., 6. The integer-pair representation of the thirteen diatonic intervals {P1, m2, M2,..., M7} is given in Fig. 1.[1] The reader may readily verify that these integer pairs convey crucial information concerning diatonic intervals, information not available in the corresponding log frequency ratios.[2]

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Perfect prime (P1)----(0,0)
Minor second (m2)-----(1,1)
Major second (M2)-----(2,1)
Minor third (m3)------(3,2)
Major third (M3)------(4,2)
Perfect fourth (P4)---(5,3)
Augmented fourth (A4)-(6,3)
Diminished fifth (d5)-(6,4)
Perfect fifth (P5)----(7,4)
Minor sixth (m6)------(8,5)
Major sixth (M6)------(9,5)
Minor seventh (m7)----(10,6)
Major seventh (M7)----(11,6)
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Figure 1. The "cognitive," integer-pair representation of diatonic intervals

As shown in Fig. 2, any diatonic interval (s,t) can be written as a product, mod (12,7), of the "perfect fourth" (5,3) and an integer n=0, ±1,..., ±6, where n · (5,3) mod (12,7) is the integer pair (5n mod 12,3n mod 7); clearly, in this cyclic representation (familiar from the traditional "cycle of fifths") the perfect fifth (7,4) can be substituted for the perfect fourth to yield the same thirteen intervals. In other words, the set of

1 A similar representation of diatonic intervals is offered in Brinkman (1986).

2 For example, the integer pairs (3,2) and (4,2) convey exactly the sense in which the diatonic intervals "minor third" and "major third" are two qualitatively different species of the same type.

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thirteen diatonic intervals which constitutes the familiar diatonic system is defined by a quintuple of integers (12,7;5,3;6) consisting of the octave (12,7), the fourth (5,3) (equivalently, the fifth 7,4), and the number 6, which defines the range from -6 to 6 through which the integer n runs. Writing the quadruple (12,7;5,3) as two fractions, namely $\frac{5}{12}$, and $\frac{3}{7}$ one sees that we have successive terms in $\mathbf{F}_{12}$, since 5•7-12•3=-1.

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d5=(6,4)  ≡ 6·(5,3) mod (12,7)
m2=(1,1)  ≡ 5·(5,3) mod (12,7)
m6=(8,5)  ≡ 4·(5,3) mod (12,7)
m3=(3,2)  ≡ 3·(5,3) mod (12,7)
m7=(10,6) ≡ 2·(5,3) mod (12,7)
P4=(5,3)  ≡ 1·(5,3) mod (12,7)
P1=(0,0)  ≡ 0·(5,3) mod (12,7)
P5=(7,4)  ≡-1·(5,3) mod (12,7)
M2=(2,1)  ≡-2·(5,3) mod (12,7)
M6=(9,5)  ≡-3·(5,3) mod (12,7)
M3=(4,2)  ≡-4·(5,3) mod (12,7)
M7=(11,6) ≡-5·(5,3) mod (12,7)
A4=(6,3)  ≡-6·(5,3) mod (12,7)
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Figure 2. Diatonic intervals as the set {(s,t)}={n(5,3) mod (12,7), n=0, ±1, . . . , ±6, 0≤s≤11, 0≤t≤6}

Before one can inquire further into the nature of this surprising connection between diatonicism and Farey series, it is necessary to bring the present, "cognitive" notion of diatonicism into sharper focus.

Suppose that instead of the specific quintuple of integers (12,7;5,3;6) which defines the familiar diatonic system we have an arbitrary quintuple (a,b;q,r;N); would {(s,t)} = {n(q,r) mod (a,b), n=0, ±1,..., ±N}, where N is a natural number, 0≤s≤a -1, 0≤t≤b -1, 1≤q≤a -1, 1≤r≤b -1, and a>b, satisfy our intuitions concerning what a "diatonic interval" is? It should not take too much effort to demonstrate that the answer is "no." The familiar diatonic system (12,7;5,3;6) has two intuitively indispensible properties which are by no means necessary properties in some arbitrary "diatonic system" (a,b;q,r;N); in Agmon (1989) these properties are termed "efficiency" and "coherence."

Efficiency is a property by which the set {(s,t)} of diatonic intervals is the **smallest** such set which contains each s=0, 1,..., a-1 at least once. Since we have an odd number (2N+1) of diatonic intervals, it follows that if a is even, one (and only one) s must appear twice in {(s,t)}; indeed, in the familiar diatonic system (a=12) we have one pair of "enharmonically equivalent" diatonic intervals satisfying this requirement, namely the inversionally related (6,3) and (6,4) (A4/d5).[3]

3 The relationship between the number 2N+1 and the integer a described above is only one of two relationships which constitute the property of efficiency; the other relationship is b=N+1. In the present context one may ignore the later relationship, the necessity of which emerges only when "scale steps"

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The second property, coherence, states that if (s,t) and (s',t') are two diatonic intervals, and if s>s', the relation t≥t' holds. Fig. 3 illustrates the property as it applies in the familiar case. The twelve integers 0, 1,..., 11 in the left hand column are integers s mod 12, and the seven integers 0, 1,..., 6 in the right-hand column are integers t mod 7. The arrows connecting the two columns establish thirteen integer-pairs (s,t) mod (12,7); these integer pairs are exactly the thirteen diatonic intervals depicted in Fig. 1.

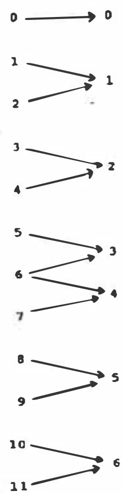

Figure 3. The correspondence s→t within the familiar set of diatonic intervals

As may be seen, when the numerical values in the left-hand column increase, the corresponding values in the right-hand column either remain the same or also increase--but never decrease. (To appreciate the non-triviality of this property recall that diatonic intervals are defined cyclically; with a cyclic definition there is no guarantee that the monotonic relationship just described between integers {s} and corresponding integers {t} be satisfied.) If one interprets an integer s as representing s semitones, and an integer t as representing the diatonic interval-type "(t+1)th" (i.e. 0="prime," 1="second,"..., 6="seventh"), the property of coherence makes the obvious claim that interval type increases monotonically (that is, either increases or stays the same, but never decreases) with the number of semitones.[4]

(and not only diatonic intervals) are taken into account. See Agmon (1989), pp. 11-13.

4 The property of coherence was first described and discussed in Balzano (1982); the term "coherence" is also borrowed from Balzano.

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(The property of efficiency may also be nicely visualized in Fig. 3; note that to each integer s mod 12 except 6 there corresponds exactly one integer t mod 7.)

It has been proven formally that in order to generate an efficient and coherent diatonic system, a, b, q, r, and N cannot be arbitrary (Agmon, 1989). In particular, if a is odd it is both necessary and sufficient not only that $b = \frac{a+1}{2}$, but also that (q,r) equal either (2,1) or its inverse, (a-2,b-1); if a is even it is necessary and sufficient that $b = \frac{a}{2} + 1$, and (q,r) equal either $(\frac{a}{2}+1, \frac{b+1}{2})$, or its inverse, $(\frac{a}{2}-1, \frac{b-1}{2})$.[5] It is not difficult to see that, in either case, we have ar - bq = ±1, and therefore $\frac{q}{a}$ and $\frac{r}{b}$ are adjacent terms in Fa. In other words, there is a necessary connection between efficient and coherent diatonic systems and Farey series; we shall say henceforth that any diatonic system (a,b;q,r;N) satisfying efficiency and coherence has the "Farey property"--FP--in the sense that a, b, q, and r satisfy the relationship ar-bq=±1.

Why do efficient and coherent diatonic systems have FP? On the basis of Agmon (1989) one can only say that while FP is a necessary property of efficient **and** coherent diatonic systems, it is not a necessary property of efficient (but not coherent) diatonic systems. Yet whether coherence is by itself a sufficient condition for FP to hold is impossible to say; in the above-mentioned source the necessary and sufficient conditions for coherence are not considered independently of those of efficiency.

As it turns out, the mathematics of coherent scale systems is quite interesting.[6] Here it is only necessary to state the following

THEOREM. *Let $DS_{(a,b;q,r;N)}$ be a coherent diatonic system defined by parameters a, b, q, **r**, and **N**, where (a,q) = 1 and (b,r) = 1. If the relation $b > N \geq \frac{a-1}{2}$ holds, then the system has FP and ar-bq=±1.*

From the theorem one leams that coherence by itself is not a sufficient condition for the Farey property to hold; but neither is it necessary that the system be efficient as well. As we have seen, efficiency requires that 2N+1 be the **smallest** number equaling at least a, that is, 2N+1=a if a is odd, or 2N+1=a+1 if a is even; this is a much stronger requirement than 2N+1≥a, which appears in the theorem.

BIBLIOGRAPHICAL CONSIDERATIONS

To the best of my knowledge, the relevance of Farey series to diatonic music in a context other than intonation was first observed by Eric Regener, in his 1973 monograph **Pitch Notation and Equal Temperament**. Following a unique yet remarkably consistent approach, Regener develops the idea of "interval space"

5 In order to arrive at these results the relationship b=N+1 must be assumed; see note 3.

6 I intend to discuss this topic in detail in a separate study.

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I--an infinite, two dimensional lattice of intervals defined by two "generators," each of which is also defined by two integers. Regener proves (Theorem 3, p. 77) that a connection exists between any two generators of I and the Farey relationship ad-bc=-1. Since Regener's formalism derives from a combination of the mod-7 properties of staff notation with the cycle of fifths, it is my conjecture that his "Farey property" and the present FP are logically equivalent; this conjecture, however, is yet to be formally proven.

In two other studies of diatonic and related scale systems the Farey relationship ad-bc=-1 makes a passing appearance; these studies are Clough and Meyerson's 1985 article "Variety and Multiplicity in Diatonic Systems," and Clough and Douthett's 1991 article "Maximally Even Sets."[7] Although these two works are based on assumptions which differ somewhat from the assumptions in Agmon (1989) and the present work, some striking correspondences between respective results nonetheless exist, as has been already noted in Agmon (1989) and Clough and Douthett (1991). As it turns out, from the mathematics of coherent scale systems one learns that the appearance of the Farey relationship in both Clough and Myerson (1985) and Clough and Douthett (1991) is equivalent to the following

CLAIM. *Let a and b be two relatively prime integers, a>b>1; let q be an integer $1 \leq q \leq a-1$ satisfying the relation $qb \equiv \pm 1$ (mod a); and let N be an integer satisfying the relation N=b-1. Then there exists a coherent diatonic system $DS_{(a,b;q,r;N)}$ uniquely determined by the relation $r=\frac{bq\mp 1}{a}$.*

MUSICAL CONSIDERATIONS

The Farey property of diatonicism is perhaps nothing but a mathematical curiosity; nonetheless, certain consequences of FP are familiar to all musicians. If FP holds and ar-bq=-1, we have immediately the following corollaries:

(1) $ar \equiv -1 \pmod{b}$;

(2) $bq \equiv 1 \pmod{a}$.

From $bq \equiv 1$ (mod a) it immediately follows that $b(q,r) \equiv (1,0)$ mod (a,b). To take the familiar diatonic system as a specific example, we have $7(7,4) \equiv (1,0)$ mod (12,7), that is, a cycle of seven perfect fifths yields an augmented prime (0,0 is a perfect prime, and therefore 1,0 is an augmented prime). Taking "C," for example, as a point of departure, after a cycle of seven fifths we land on a "C#." Although this

7 In Clough and Myerson 1985 (Lemma 4 on p. 263) we have d'=(cc'+1)/d; in Clough and Douthett 1991 (proof of Theorem 3.1 on p. 148) we have $cl=dg_1+1$.

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chromatic property of a cycle of seven fifths is wellknown, it is probably not so wellknown that it follows from FP.

From ar≡-1 (mod b) it immediately follows that a(q,r)≡(0,b-1) mod (a,b). Again, taking the familiar diatonic system as an example, we have 12(7,4)≡(0,6) mod (12,7), that is, a cycle of twelve fifths yields an augmented seventh (11,6 is a major seventh, and therefore 0,6 is an augmented seventh). Starting from "C" once again, the note twelve fifths away is the **enharmonically equivalent** "B#."

Note that the relation 12(7,4)≡(0,6) mod (12,7) says nothing concerning the relationship between C and B# in terms of log frequency ratios. It is, of course, not a mere coincidence that a cycle of twelve pure "perfect fifths" in the sense of $12 \cdot \log_2 1.5$ modulo the "perfect octave" (that is, mod 1) gives a very good approximation of the pure "perfect prime" (that is, of 0). Indeed, as has been shown in Agmon (1989, pp. 20-21), the familiar diatonic system, mod (12,7), is specifically selected among all **possible** diatonic systems, cognitively defined, precisely for the sake of this remarkable correspondence between the "perfect fifth" in its cognitive sense and the "perfect fifth" in its perceptual sense (i.e., the log frequency-ratio 3/2). Unfortunately, it is still very common to assume that the two kinds of fifths are identical, rather than to see them as essentially different, albeit related, diatonic phenomena.

CONCLUSION

John Farey wrote several articles on systems of intonation (Farey 1807; 1810; 1811; cited in Regener 1973), yet apparently did not see a connection between his work on music and the numerical series that bear his name (Farey, 1816). The relevance of Farey series to music, however, transcends the limited question of diatonic intonation. As we have seen, there exists a necessary connection between diatonicism and Farey series at a more abstract, "cognitive" level. This connection is in a sense a by- product of certain other properties of diatonicism, notably the intuitively essential property of "coherence." However, as we have also seen, the "Farey property" of diatonicism has musical consequences all of its own, which concern the theory of chromaticism and enharmonicism.

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Supplementary Sets - Theory and Algorithms

Dan Tudor Vuza
The Institute of Mathematics of the Romanian Academy

0. Introduction

One of the most interesting situations encountered in the study of pitch class sets is represented by the existence of partitions of the set of all pitch classes into subsets which belong to the same transpositional class. For instance, there exists:

a) a partition into four augmented trichords;

b) a partition into three diminished-seventh tetrachords;

c) a partition into three tetrachords {B, C, F, G♭}, {C#, D, G, A♭}, {D#, E, A, B♭};

d) a partition into three minor tetrachords {C, D, E♭, F}, {E, F#, G, A}, {G#, A#, B, C#};

e) a partition into four trichords {B, C#, D#}, {C, D, E}, {F, G, A}, {G♭, A♭, B♭}.

The theory of such partitions was examined by the author in his paper [4]. A transpositional class (or type) with the property that there is a partition of the set of all twelve pitch classes into subsets belonging to that class was called there a *partitioning class*. A glance at examples *a*) - *e*) might leave the impression that there is not an apparent relation between the phenomenon of partitioning classes and the phenomenon of transpositional symmetry: examples *a*) - *c*) involve classes with transpositional symmetry, while the classes in examples *d*) - *e*) do not possess such a symmetry. It was demonstrated in [4] that there is however a close connection between the two phenomena. To this end the notions of supplementary sets and of supplementary classes were introduced. Two pitch-class sets M and N have been called *supplementary* if the product of their number of elements equals 12 and the intersection between the set of intervals spanned by the elements in M and the set of intervals spanned by the elements in N is reduced to the null interval. Equivalently, the collection of all transpositions of M by the intervals from, say, the pitch-class C to

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