Supplementary Sets - Theory and Algorithms

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0. Introduction

One of the most interesting situations encountered in the study of pitch class sets is represented by the existence of partitions of the set of all pitch classes into subsets which belong to the same transpositional class. For instance, there exists:

a) a partition into four augmented trichords;

b) a partition into three diminished-seventh tetrachords;

c) a partition into three tetrachords {B, C, F, G♭}, {C#, D, G, A♭}, {D#, E, A, B♭};

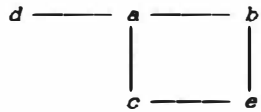
d) a partition into three minor tetrachords {C, D, E♭, F}, {E, F#, G, A}, {G#, A#, B, C#};

e) a partition into four trichords {B, C#, D#}, {C, D, E}, {F, G, A}, {G♭, A♭, B♭}.

The theory of such partitions was examined by the author in his paper [4]. A transpositional class (or type) with the property that there is a partition of the set of all twelve pitch classes into subsets belonging to that class was called there a *partitioning class*. A glance at examples *a*) - *e*) might leave the impression that there is not an apparent relation between the phenomenon of partitioning classes and the phenomenon of transpositional symmetry: examples *a*) - *c*) involve classes with transpositional symmetry, while the classes in examplee *d*) - *e*) do not possess such a symmetry. It was demonstrated in [4] that there is however a close connection between the two phenomena. To this end the notions of supplementary sets and of supplementary classes were introduced. Two pitch-class sets M and N have been called *supplementary* if the product of their number of elements equals 12 and the intersection between the set of intervals spanned by the elements in M and the set of intervals spanned by the elements in N is reduced to the null interval. Equivalently, the collection of all transpositions of M by the intervals from, say, the pitch-class C to

each pitch-class in N forms a partition of the set of twelve pitch classes. Two transpositional classes have been called *supplementary* if they are respectively the classes of two supplementary sets.

The notion of supplementary classes is related to that of a partitioning class by the reeult asserting that a class M is partitioning iff (= if and only if) there is a class N so that M and N are supplementary. For instance, the diagram below exhibits the pairs of supplementary classes in examples *a*) - *e*).



The connection with transpositional symmetry is now apparent due to the following theorem [4]:

THEOREM 0.1. *Given any pair of supplementary sets, at least one set in the pair has transpositional symmetry.*

This theorem has of course both theoretical and practical consequences. The strong restrictions it imposes on the possibilities of finding partitioning classes allowed the author to work out the complete list of pairs of supplementary classes in a relatively small number of steps (see [4]).

It is by now a well-founded principle that the mathematics of the sets of residue classes may serve to model phenomena in the universe of pitch-class sets as well as rhythmic phenomena characterized by periodicity; the first systematic applications of it go back to Babbit [1]. In accord with this general principle, I have tried to answer in my study in four parts [6] the following questions:

QUESTION 1. What is the rhythmic analog of a partition into subsets belonging to the same transpositional class? What is the rhythmic analog of supplementary classes?

QUESTION 2. What is the rhythmic interpretation of the fact that one of the classes in a supplementary pair has transpositional symmetry?

QUESTION 3. Does Theorem 0.1 remain true when the group $\mathbb{Z}_{12}$ is replaced by a group $\mathbb{Z}_n$ for some arbitrary integer n?

In Babbit´s model one considers a set T of regular pulses (successive pulses separated by the same time interval which is taken as time unit) and one sets a correspondence between the subsets of $\mathbb{Z}_{12}$ and those periodic subrhythms of of T whose periods count 1, 2, 3, 4,

6, or 12 time units. By applying this procedure to a partition of $\mathbb{Z}_{12}$ into subsets belonging to the same transpositional class one obtains a partition of the total rhythm T into periodic subrhythms $R_1, \ldots, R_l$ so that for any couple i, j the rhythm R_i is obtained from R_j via a temporal translation. We analyze separately the musical significance of the properties of the rhythmic partition $R_1, \ldots, R_l$. In the course of this analysis we suppose that each R_i represents the set of time points associated with the rhythmic pattern delivered by a voice V_i.

The fact that the R_i's can be obtained each from the other via a temporal translation means that the voices $V_1, \ldots, V_l$ all together are singing a rhythmic canon in strict style; the fact that each R_i is periodic means that the canon is unending.

The fact that the R_i's form a partition of the total rhythm means on the one hand that there are no beats (attacks) in common among different voices (so that the voices are "complementary") and on the other hand that the resultant rhythm obtained by adding up the beats from all voices equals the total rhythm. In other words, every pulse in the total rhythm corresponds to a beat from one and only one voice. Rhythmic canons of the type described above were called in the mentioned study *regular complementary unending canons*.

To find the rhythmic analog of supplementary classes, consider first the situation when the voices $V_1, \ldots, V_l$ are singing an arbitrary unending rhythmic canon in strict style and let R_i be the set of time points associated with the voice V_i. We express the fact that each R_i represents the same rhythmic pattern except for a temporal translation by saying that all the sets R_i belong to the same rhythmic class R, which we have called the *ground class* of the canon in question. Suppose now that the beginning of each period of R_i is marked by a metric accent. If one adds together the metric accents from all voices one obtains another periodic rhythm, whose rhythmic class S is referred to as the *metric class* of the canon. In the mentioned study the notion of supplementary rhythmic classes was defined and it was shown that an unending rhythmic canon is regular and complementary iff its ground class and its metric class are supplementary.

To answer Question 2, observe that the metric class controls the relative distances in time between the voices in a canon. The ratio

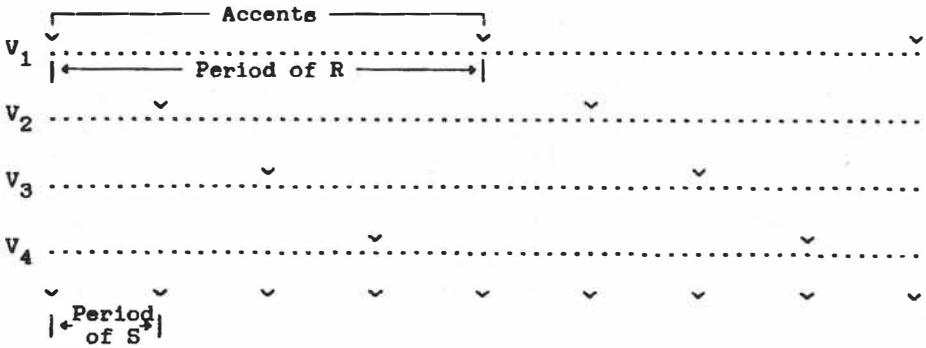
$$\frac{\text{Period of } R}{\text{Period of } S}$$

is an integer which divides the number l of voices, and hence can take values only in the range from 1 to l. (The period of a rhythmic class

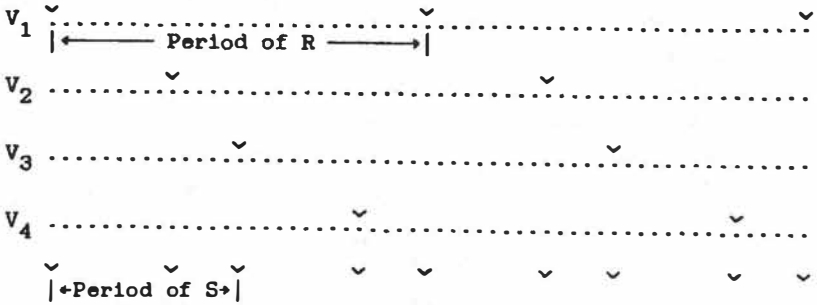
is defined as the period of any periodic rhythm belonging to that class). If that ratio equals *1*, the relative dietances divide the period of *R* into *1* equals parts (see Example 0.1).

If the ratio equals *1/2*, we have a grouping of the relative dietances which is repeated every half a period of *R* (see Example 0.2).

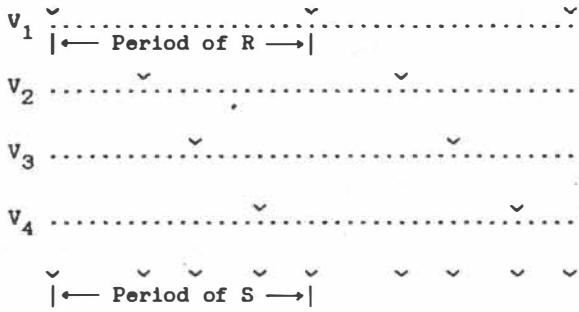
Similar groupings can be found if the ratio in question is not 1. If however the ratio equals 1, there is no grouping of the relative distances which is regularly repeated within a period of *R* (see Example 0.3).



Example 0.1



Example 0.2



Example 0.3

Canons of the latter type will be referred in the following as *canons of maximal category*. The conclusion of the above discussion is that the statement that one transpositional class in a certain pair of supplementary classes has transpositional symmetry corresponds, in the rhythmic domain, to the statement that a certain regular complementary canon is not of maximal category.

We come now to Question 3. If we confined ourselves to canons obtained via rhythmic interpretation of subsets of $\mathbb{Z}_{12}$, then according to Theorem 0.1, we would never obtain regular complementary unending canons of maximal category. A higher level of generality in the study of regular complementary canons is attained by employing the model for periodic rhythm proposed by the author in the study [5], which may be regarded as an extension and a further elaboration of Babbit´s ideas. In this model, various rhythmic classes correspond to translation classes of *different* groups $\mathbb{Z}_n$, the integer n ranging from 1 to infinity (a translation class is the $\mathbb{Z}_n$-analog of what we used to call, for $n = 12$, a tranepositional class). More explicitly, if the rhythmic classes R_1 and R_2, correspond to a translation class of $\mathbb{Z}_{n_1}$ and $\mathbb{Z}_{n_2}$, respectively, then certain set-theoretic operations applied to periodic rhythms belonging to R_1 and R_2 may lead to rhythms whose classes correspond to translation classes of $\mathbb{Z}_{n_3}$, the integer n_3 possibly taking arbitrary large values even if n_1 and n_2 do not vary.

Regular complementary unending canons are still related, from the viewpoint of the extended rhythmic model, to supplementary rhythmic classes, while the latter are closely related to supplementary subsets of $\mathbb{Z}_n$ (their definition is formally the same ass previously with the only difference being that 12 is replaced by n). The conclusion is that the study of the canons of the mentioned type motivated the study

of supplementary subsets of $\mathbb{Z}_n$ for n an arbitrary positive integer. For this reason it was important to answer Question 3. The answer was negative and it was in fact possible to describe explicitly the set of those integers n with the property that Theorem 0.1 is true for all pairs of supplementary subsets of $\mathbb{Z}_n$.

From another viewpoint, we may remark that the number of elemente in at least one set in a pair of supplementary subsets of $\mathbb{Z}_{12}$ has the form p^k with p a prime (2 or 3 for $\mathbb{Z}_{12}$). We may therefore ask whether it is true that at least one set in a pair of supplementary subsets of $\mathbb{Z}_n$ has "transpositional" symmetry every time when the number of elements in at least one set in that pair has the form p^k with p a prime. It turns out that this is indeed true for all values ≥ 2 of n.

The theorems about supplementary sets are given an interpretation within the context of the theory of canons. For instance, the discovery of the fact that Theorem 0.1 is not true for certain values of n has the consequence that *there exist regular complementary unending canons of maximal category*. The signification of those theorems is at least twice. On the one hand, they impose strong restrictions on the construction of regular complementary unending canons. As an illustration, no specialist in rhythmic counterpoint could construct a regular complementary unending canon of maximal category on 2, 3, 4, 5, 7, 8, 9 or p^k voices (p a prime, k an integer ≥ 1) for the reason that this is a mathematical impossibility! On the other hand, those theorems may be regarded in connection with the general principle according to which the simpler the arithmetical structure of a number measuring a certain musical phenomenon, the clearer is the structure of that phenomenon. In our specific situation, it is true that whenever the arithmetical structure of the integers measuring the complexity of a regular complementary canon is simple (in the sense that there are not too many primes dividing them), then the canon itself has a neat and clear (at least from the mathematical viewpoint) recursive structure, as it can be obtained by successive applications of a very neat procedure described in the following under the name "elementary derivation".

Regular complementary unending canons of simple structure often occur in the rhythmic organization of musical works from the preclassical period. Canons of the type mentioned and of more complicated structure seem not to have been used so far, as their construction looks quite difficult without the aid of a mathematical theory. It was the purpose of our study [6] to lay the foundations of such a theory and to demonstrate that even in connection with this polyphonic form

subjected to such strong mathematical restrictions, the composer has quite a large choice of constructive and transformational procedures which may be combined in a creative way.

From the mathematical viewpoint, the theory of supplementary sets represents an instance of application of Fourier analysis of finite groups to music theory, thus continuing a line of research inaugurated by Professor David Lewin in 1959. In fact, the entire theory of supplementary sets can be subsumed to a problem area indicated by Lewin on page 103 of his remarkable book [3]. The problems arising in this area are hard, so that technical difficulties are often unavoidable when trying to solve them. Music is an extremely complex phenomenon; we therefore expect the mathematics used in studying it to support a similar complexity. With this principle in mind, the author decided to present to the readers of *Perspectives of New Music* a detailed account of his own research, including the mathematical proofs which spread across several pages. After one year since the last part of the named study [6] was published, it seemed that its technical complexity represented a difficulty against its correct perception by the musical community. By writing that study, the author also hoped to make one step forward in the direction of interdieciplinary collaboration in a fair spirit and from equal positions, and he was quite surprised to be not understood in his intentions. The present contribution is thus an attempt to make somewhat more accessible the work in [6] by retaining from it only the main concepts and procedures. We hope that in this way, our work would, as Professor John Rahn says, "be useful to or inspirational for musical thinkers or doers of the future".

1. The rhythmic model

The model of periodic rhythm proposed by the author of the present article was exposed in the study [5]. The following is only a brief reminder of the ooncepts introduced by that model which are needed for our presentation. The reader is referred to the referenced study for a detailed discussion.

First of all, some notations to be used in connection with the subsets of a commutative group G. If R_1, R_2 are two such subsets and t is any element of G, then

$$t + R_1 = \{t + r \mid r \in R\},$$
$$R_1 + R_2 = \{r_1 + r_2 \mid r_1 \in R_1,\ r_2 \in R_2\},$$

$$R_1 - R_2 = \{r_1 - r_2 \mid r_1 \in R_1,\ r_2 \in R_2\}.$$

The subsets R_1, R_2 of G are said to belong to the same translation class if there is $t \in G$ such that $R_1 = t + R_2$. The translation class of a subset R is denoted by [R]. The set of all translation classes of G is denoted by $T(G)$.

$\mathbb{Q}_+$ will be the set of all strictly positive rational numbers. Recall that $a \in \mathbb{Q}_+$ is said to divide $b \in \mathbb{Q}$ (in symbols, $a|b$) if b/a is an integer. For any subset R of $\mathbb{Q}$ and any $a \in \mathbb{Q}_+$ we let aR be the set $\{ar \mid r \in R\}$.

DEFINITION 1.1. *A periodic rhythm is a (possibly empty) subset* R *of the set* $\mathbb{Q}$ *of rational numbers satisfying* (R1) *and* (R2) *below:*

(R1) $t + R = R$ for some $t \in \mathbb{Q} \setminus \{0\}$ (in other words, R is a periodic subset of the additive group $\mathbb{Q}$);

(R2) For every a, $b \in \mathbb{Q}$ with $a < b$, the set $R \cap [a, b)$ is finite.

We have denoted by $[a, b)$ the set $\{x \mid x \in \mathbb{Q},\ a \leq x < b\}$.

The elements in R should be viewed as marking the transition moments ("beats") from one musical event to another during the discourse delivered by a single voice. (Hence the beginning of a pause is also marked by an element in R).

As we shall deal only with periodic rhythms in the sense of the above definition, we will by convention omit the adjective "periodic" and speak simply about "rhythms".

The fact that the time axis in the rhythmic model is $\mathbb{Q}$ corresponds to the reality that in European music all durations are denoted by rational numbers. Besides, this fact provides a lot of formal advantages: for instance, any finite union and any finite intersection of rhythms is still a rhythm.

As examples of rhythms let us consider

$$R_1 = \{0,\ 3/8,\ 1/2\} + (3/4)\mathbb{Z},$$
$$R_2 = \{1/8,\ 3/16,\ 9/16,\ 11/16,\ 3/4,\ 13/16,\ 7/8,\ 15/16\} + \mathbb{Z},$$
$$R_3 = \{0,\ 1/16,\ 1/8,\ 3/16,\ 1/4,\ 7/16,\ 1/2,\ 7/8\} + \mathbb{Z}.$$

We have for instance

$$R_1 \cap R_2 = \{-9/8,\ -1/4,\ 3/4,\ 9/8\} + 3\mathbb{Z},$$
$$R_2 \cap R_3 = \{1/8,\ 3/16,\ 7/8\} + \mathbb{Z}.$$

DEFINITION 1.2. *A rhythmic class is the translation class (with respect to the additive group* $\mathbb{Q}$*) of a rhythm.*

To distinguish between rhythms and rhythmic classes we shall use capital Roman letters for the former and capital italics for the latter. The rhythmic class of the rhythm R will be denoted by [R]. The notation *Rhyt* will be used for the set of all rhythmic classes.

The following numerical entities are attached to a rhythm R:

a) the *period* Per R of R, defined as the least $t \in \mathbb{Q}_+$ satisfying $t + R = R$;

b) the *minimal division* Div R of R, defined as the greatest $d \in \mathbb{Q}_+$ which divides every element in $R - R$;

c) the *number of attacks per period* Nrp R of R, defined as the number of elements in $R \cap [a, a + \text{Per } R)$ (this number does not depend on the particular choice of $a \in \mathbb{Q}$).

For our previous example we have

$$\text{Per } R_1 = 3/4, \quad \text{Div } R_1 = 1/8, \quad \text{Nrp } R_1 = 3$$
$$\text{Per } R_2 = 1, \quad \text{Div } R_2 = 1/16, \quad \text{Nrp } R_2 = 8.$$

Clearly enough, if $[R_1] = [R_2]$ then $\text{Per } R_1 = \text{Per } R_2$, $\text{Div } R_1 = \text{Div } R_2$ and $\text{Nrp } R_1 = \text{Nrp } R_2$. We may therefore define (by use of representants) the corresponding numerical entities Per *R*, Div *R* and Nrp *R* for a rhythmic class *R*.

DEFINITION 1.3. *A regular rhythm is a rhythm of the form* $a + t\mathbb{Z}$ *with* $a \in \mathbb{Q}$, $t \in \mathbb{Q}_+$. *A regular class is the rhythmic class of a regular rhythm.*

Hence regular classes are the rhythmic classes of the form $[t\mathbb{Z}]$ with $t \in \mathbb{Q}_+$. In order to simplify the notation we make the convention of writing $[t]$ instead of $[t\mathbb{Z}]$.

DEFINITION 1.4. *Let* k *be any integer* ≥ 1. *A meter of order* k *on a rhythm* R *is a regular rhythm* S *satisfying* $S \subset R$ *and* $\text{Per } S = k \text{ Per } R$.

The rhythm S should be viewed as marking the "strong" beats in R.

DEFINITION 1.5. *Two rhythmic classes R, S are called intervallically disjoint if* $(R - R) \cap (S - S) \subset (\text{Per } R \vee \text{Per } S)\mathbb{Z}$ *for any representants* $R \in R$ *and* $S \in S$.

By $a \vee b$ we have denoted the least common multiple of $a, b \in \mathbb{Q}_+$

In other words, as both R and S repeat over and over indefinitely, the only temporal intervals one can form using time points both from R and also from S, are exactly those temporal intervals that are common multiples of Per R and Per S.

We use the symbol $R \perp S$ for the situation described by the above

definition.

The semigroup structure of *Rhyt* represents the main algebraic structure on that set which proved itself to be a useful device in studying the construction of canons. The semigroup structure is defined by means of a composition law whose definition is as follows: if R, $S \in$ *Rhyt*, then their composition $R + S$ equals [R + S], where R and S, respectively is any rhythm in R and S, respectively.

DEFINITION 1.6. *Let R, $S \in$ Rhyt. We say that R is a condensation of S, or that S is an extension of R if there is $t \in \mathbb{Q}_+$ such that $S + [t] = R$.*

In symbols, we write $S \longrightarrow R$ for "R is a condensation of S".

In order to present musical examples it is important to have a practical method of labelling the rhythmic classes. This is achieved by the use of intervallic structures. An *intervallic structure with m elements* is any non-periodic sequence $s_1, \ldots, s_m$ in $\mathbb{Q}_+$ (i.e., the identity map is the unique cyclic permutation of $\{1, \ldots, m\}$ leaving the sequence invariant). Such a sequence is said to correspond to a rhythmic class R if there are R $\in R$ and $t \in$ R so that $s_i = t_{i+1} - t_i$ for $1 \leq i \leq m$, where $t = t_1 < t_2 < \ldots < t_m$ are the elements in R $\cap$ [t, t + Per R) and $t_{m+1} = t$ + Per R. In other words, an intervallic structure records the intervals between those successive attacks in a rhythm R which lie inside an interval of length Per R spanned by two attacks in R.

To every intervallic structure $s_1, \ldots, s_m$ there corresponds a unique rhythmic class, which we shall denote by $[s_1, \ldots, s_m]$. In the converse direction, to one rhythmic class there correspond several intervallic structures; however, each of these can be obtained from any other via a cyclic permutation. Hence, if one call equivalent two intervallic structures which differ only by a cyclic permutation, one may assert that *there is a one-to-one correspondence between rhythmic classes and classes of equivalent intervallic structures.*

As outlined in the Introduotion, there are bijective correspondences between the sets of translation classes $T(\mathbb{Z}_n)$ of $\mathbb{Z}_n$ and certain subsets of *Rhyt* (n is here ranging over all integers ≥ 1, so we agree that $\mathbb{Z}_1 = \{0\}$). We recall their formal construction (see [5] for details). For every pair $(a,b) \in \mathbb{Q}_+ \times \mathbb{Q}_+$ such that $a|b$, let us denote by $Rhyt_{a,b}$ the set of all $R \in$ *Rhyt* satisfying $a|$Div R and Per $R|b$. The

collection of the $Rhyt_{a,b}$, as (a,b) ranges over $\mathbb{Q}_+ \times \mathbb{Q}_+$, is an upwards directed collection of sets whose union equals $Rhyt$. The bijection $H_{a,b} : Rhyt \longrightarrow T(\mathbb{Z}_n)$, where $n = b/a$, is obtained as follows. By definition, $R \in Rhyt_{a,b}$ if R is the class of a rhythm R verifying $a^{-1}R \subset \mathbb{Z}$ and $b + R = R$. Then let $H_{a,b}(R)$ be the translation class of the set of residue classes modulo n of the elements in $a^{-1}R$.

Let $R = [s_1, \ldots, s_m]$ be a rhythmic class in $Rhyt_{a,b}$. To compute $H_{a,b}(R)$ let $t_1, \ldots, t_{km}$ be the sequence obtained by repeating $s_1, \ldots, s_m$ k times, where $k = b/Per\ R$. Since $R \in Rhyt_{a,b}$ the numbers $a^{-1}t_1, \ldots, a^{-1}t_{km}$ are integers, so that we can consider their residue classes $r_1, \ldots, r_{km}$ modulo $n = b/a$. Then $H_{a,b}(R)$ is the translation class in $\mathbb{Z}_n$ of the set of partial sums

$$\sum_{j=1}^{i} \{r_j | 1 \leq i \leq km\}.$$

Conversely, given a subset M of $\mathbb{Z}_n$, consider a sequence $t_1 < t_2 < \ldots < t_m$ of integers in $[0,n)$ such that their residue classes modulo n form precisely the set M. Let $r_1, \ldots, r_q$ be the sequence defined by $r_i = t_{i+1} - t_i$ for $1 \leq i \leq q-1$, $r_q = n + t_1 - t_q$. If that sequence is non-periodic, then $H^{-1}_{a,b}([M])$ equals $[r_1, \ldots, r_q]$. Otherwise, $H^{-1}_{a,b}([M])$ equals $[s_1, \ldots, s_m]$, where $s_1, \ldots, s_m$ is obtained by taking the first m terms from $r_1, \ldots, r_q$; m is uniquely determined from the requirements that $s_1, \ldots, s_m$ should be non-periodic and by repeating it q/m times we should obtain $r_1, \ldots, r_q$.

2. Unending canons from the viewpoint of the rhythmic model

In an unending rhythmic canon in strict style, two or more voices starting at different points in time deliver the same rhythmic pattern. Because of the strict identity between the rhythmic patterns produced by the different voices, it follows that for any couple V_1, V_2 of voices in the canon, the attacks of V_1 are separated by a constant temporal interval t_{12} from the attacks of V_2. Moreover, the pattern is periodically repeated by each voice. And finally, since we consider here only rhythmic aspects and neglect others such as pitch, we may and shall assume that there is no complete overlapping of attacks between any two different voices in the canon.

Within the framework of the rhythmic model of Section 1, the above described musical situation is abstracted in the following definit-

..., S_l} *of regular rhythms satisfying the conditions below:*

(M1) *For any* $i \in \{1, \ldots, l\}$, S_i *is a meter of order* k *on* R_i;

(M2) *For any couple* $i, j \in \{1, \ldots, l\}$, *there is* $t_{ij} \in \mathbb{Q}$ *such that* $R_i = t_{ij} + R_j$ *and* $S_i = t_{ij} + S_j$.

Remark that the existence of t_{ij} such that $R_i = t_{ij} + R_j$ is implied by the very definition of a canon, while the existence of t'_{ij} such that $S_i = t'_{ij} + S_j$ is a consequence of the equality of the periods of S_i and S_j. The identity between the relations "S_i to R_i" and "S_j to R_j" is expressed in Definition 2.3 precisely by the identity $t_{ij} = t'_{ij}$.

In addition of hearing the voices in a canon as embedded into a resultant structure, we are also interested in hearing the metric structures as a whole; in other words, if {S_1, ..., S_l} is a meter on $\mathfrak{C}$, we are interested in the resultant of the meter, that is, the superposition of the metric accents from all voices. We say therefore that $\mathfrak{C}$ admits a rhythmic class S as *a metric class of order* k if there is a meter {S_1, ..., S_l} of order k on $\mathfrak{C}$ such that

$$S = [\bigcup_{i=1}^{l} S_i].$$

It can be proved that there is a unique rhythmic class admitted by $\mathfrak{C}$ as a metric class of order 1; we shall call it the *primary metric class* of $\mathfrak{C}$ and denote it by *Met* $\mathfrak{C}$. On the contrary, if $k \geq 2$ then $\mathfrak{C}$ admits in general several metric classes of order k, which we call *secondary metric classes*.

We list now the integers which, as announced in the Introduction, serve as numerical measures of the complexity of a canon $\mathfrak{C}$:

a) the *ground number* of $\mathfrak{C}$, defined as Nrp *Grd* $\mathfrak{C}$;

b) the *category* of $\mathfrak{C}$, defined as Nrp *Met* $\mathfrak{C}$;

c) the *modulus* of $\mathfrak{C}$, defined as Per *Grd* $\mathfrak{C}$ / Div *Res* $\mathfrak{C}$.

All these are absolute invariants, that is, they do not change when one makes changes in the choices of the referential time-point zero and the referential time unit.

The reason behind the term "modulus" should be clear from Section 1: a canon of modulus n involves translation classes of $\mathbb{Z}_n$, namely the classes $H_{a,b}$(*Grd* $\mathfrak{C}$) and $H_{a,b}$(*Met* $\mathfrak{C}$), with a = Div *Res* $\mathfrak{C}$, b = Per *Grd* $\mathfrak{C}$ and $n = b/a$.

Concerning the category, this terminology is inspired from Grigoriev and Müller [2]. In Section 48 of that book, a canon $\mathfrak{C}$ on two voices is said to be of first category if (in our terminology), Nrp

ion.

DEFINITION 2.1. *An unending rhythmic canon is a finite non-empty subset of a rhythmic class.*

In the spirit of the convention of Section 1, we shall omit the words "unending" and "rhythmic" and speak simply about "canons". Canons will be designated by the letter $\mathcal{C}$. Thus, if $\mathcal{C} = \{R_1, \ldots, R_l\}$ is a canon, R_i represents the set of time points associated with the i-th voice in the canon, while l equals the *number of voices* in the canon $\mathcal{C}$. For formal reasons and in accord with Definition 2.1, we shall be obliged to consider also as canons those sets $\mathcal{C}$ consisting of a single rhythm.

DEFINITION 2.2. *Two canons* $\{R_1, \ldots, R_l\}$ *and* $\{R_1', \ldots, R_l'\}$ *are called equivalent if* $\{R_1', \ldots, R_l'\} = \{t + R_1, \ldots, t + R_l\}$ *for some* $t \in \mathbb{Q}$.

An equivalence class with respect to the above introduced relation is called a canon class. Hence, equivalent canons may be viewed as obtained each from the other via a temporal translation. In particular, the number of voices in equivalent canons must be the same.

There are several rhythmic classes connected to a canon $\mathcal{C} = \{R_1, \ldots, R_l\}$. Firstly, there is the *ground class Grd* $\mathcal{C}$, which is by definition the rhythmic class implied by Definition 2.1; that is,

$$Grd\ \mathcal{C} = [R_1] = \ldots = [R_l].$$

An equality $Grd\ \mathcal{C} = R$ will also be referred to as "$\mathcal{C}$ is built on R". Secondly there is the *resultant class Res* $\mathcal{C}$, defined as

$$Res\ \mathcal{C} = [\bigcup_{i=1}^{l} R_i] \ ;$$

it is the class of the rhythm perceived by hearing the canon as a whole, that is, by superposing the attacks of all voices involved in it.

To assign other rhythmic classes to the canon $\mathcal{C}$, we have to consider the problem of meters on canons. Suppose that a meter S_i in the sense of Definition 1.4 is imposed on each rhythm R_i in $\mathcal{C}$. In order that strict identity between the rhythmic patterns delivered by the voices in the canon be observed, we have to impose the condition that for any couple i, j, the meter S_i bears to R_i the same relation as S_j does to R_j. This requirement is formally expressed by the following definition.

DEFINITION 2.3. *Let* $\mathcal{C} = \{R_1, \ldots, R_l\}$ *be a canon on* l *voices (so that* $R_i \neq R_j$ *for* $i \neq j$*). A meter of order* $k \geq 1$ *on* $\mathcal{C}$ *is a set* $\{S_1,$

Met $\mathfrak{C}$ = 1, of second category if Nrp *Met* $\mathfrak{C}$ = 2.

The following proposition establishes the fundamental relations among the above described entities attached to a canon.

PROPOSITION 2.1. *For any canon* $\mathfrak{C}$ *the following are true:*

i) *Res* $\mathfrak{C}$ = *Grd* $\mathfrak{C}$ + *Met* $\mathfrak{C}$;

ii) Per *Met* $\mathfrak{C}$ | Per *Grd* $\mathfrak{C}$;

iii) *The number of voices in* $\mathfrak{C}$ *equals the product between the category of* $\mathfrak{C}$ *and the ratio* Per *Grd* $\mathfrak{C}$ / Per *Met* $\mathfrak{C}$.

Recall that in *i)* above, "+" denotes the composition of rhythmic classes as described in Section 1.

As a consequence of *i)* we have that Div *Res* $\mathfrak{C}$ = Div *Grd* $\mathfrak{C}$ $\wedge$ Div *Met* $\mathfrak{C}$, where $a \wedge b$ denotes the greatest common divisor of $a, b \in \mathbb{Q}_+$.

It follows in particular from the above proposition that for a given number l of voices, the category must divide l and hence is restricted to the range 1, ..., l. This remark motivates the following definition.

DEFINITION 2.4. *A canon of maximal category is a canon whose category equals its number of voices.*

It also follows that the higher is the category, the less is the ratio Per *Grd* $\mathfrak{C}$ / Div *Res* $\mathfrak{C}$. Category one corresponds to evenly distributed primary metric accents (or in other words, the l voices are successively entering the canon at equally spaced moments separated by the same interval equal to $1/l$ of the period of the canon). Maximal category corresponds to the situation when the period of the primary meter equals the period of the canon (or in other words, no repetition of the resultant metric pattern can be observed within one period of the canon).

In order to give musical examples we must dispose, as in the case of rhythmic classes, of a method for labelling canon classes. To this purpose we start with the following construction. Let $R, S \in$ *Rhyt* be given. Choose the representants $R \in R$, $S \in S$. The collection of all sets of the form $s + R$, as s ranges over S, forms a canon $\mathfrak{C}$ (it is indeed a finite collection). The canon class of $\mathfrak{C}$ does not depend on the particular choices of representants; we denote it by *Can*(R, S).

By a *normal pair* we mean any ordered pair (R, S) of rhythmic classes such that Per S | Per R. It can be then proved that *in order that there is a (necessarily unique up to equivalence) canon* $\mathfrak{C}$ *built on* R *and admitting* S *as its primary metric class, it is necessary and*

sufficient that (R, S) *be a normal pair, in which situation the class of* $\mathfrak{C}$ *equals* $Can(R, S)$. *Hence the map* $\mathfrak{C} \longrightarrow (Grd\ \mathfrak{C}, Met\ \mathfrak{C})$ *induces a bijective correspondence between canon classes and normal pairs.*

DEFINITION 2.5. *Let* $Can(R, S)$ *and* $Can(R', S')$ *be two canon classes represented with the aid of the normal pairs* (R, S) *and* (R', S'). *We say that* $Can(R, S)$ *is a condensation of* $Can(R', S')$ *if* $R \longrightarrow R'$ *and* $S \longrightarrow S'$.

The relation "$Can(R, S)$ is a condensation of $Can(R', S')$" is a partial order on the set of canon classes.

DEFINITION 2.6. *The minimal condensation of maximal category (minmax condensation in short) of a canon class is the least element (for the above introduced order relation) in the set of all condensations of that class whose categories are maximal.*

In other words, if $Can(R_0, S_0)$ is the minmax condensation of $Can(R, S)$, then it can be characterized as the "less condensed form" of $Can(R, S)$ whose category is maximal (i.e., if $Can(R_1, S_1)$ is any condensation of $Can(R, S)$ whose category is maximal, then necessarily $Can(R_1, S_1)$ is a condensation of $Can(R_0, S_0)$).

It can be shown that the minmax condensation indeed exists and that we have the following algorithm for computing it: start with some normal pair (R, S). Define inductively the sequence (R_n, S_n) for $n \geq 0$ via

$$R_0 = R,\ S_0 = S,$$
$$R_{n+1} = S_n,\ S_{n+1} = R_n + [\mathrm{Per}\ S_n].$$

There exists an even integer n_0 (depending on the starting pair) such that $\mathrm{Per}\ R_{n_0} = \mathrm{Per}\ S_{n_0}$. The class $Can(R_{n_0}, S_{n_0})$ is precisely the minmax condensation of $Can(R, S)$.

As an illustration, consider the following canon $\mathfrak{C}$ on twelve voices endowed with a secondary meter of order 2 (the letter "o" denotes the attacks of each voice, while the distance between two dots is supposed to equal Div *Res* $\mathfrak{C}$; the last line represents the resultant of the meter).

The ground class is [1/4, 5/16, 1/4, 11/16] while the secondary meter class is [7/16, 5/16, 1/8, 1/8]. To determine the primary metric class simply apply a condensation by the ground number to the secondary metric class. The latter equals 3/2, so

$$Met\ \mathfrak{C} = [7/16,\ 5/16,\ 1/8,\ 1/8] + [3/2] = [1/4,\ 1/8,\ 1/16,\ 1/16].$$

Since Div *Res* $\mathfrak{C}$ = Div *Grd* $\mathfrak{C}$ $\wedge$ Div *Met* $\mathfrak{C}$ = 1/16, the modulus equals 24. The category is 4. The ratio Per *Grd* $\mathfrak{C}$ / Per *Met* $\mathfrak{C}$ equals 3, which multiplied by the category gives 12, the number of voices. To compute the minmax condensation we start with R_0 = *Grd* $\mathfrak{C}$, S_0 = *Met* $\mathfrak{C}$ and form successively

$$R_1 = S_0,\quad S_1 = R_0 + [\text{Per } S_0] = [3/16,\ 1/16],$$
$$R_2 = S_1,\quad S_2 = R_1 + [\text{Per } S_1] = [1/8,\ 1/16,\ 1/16].$$

Hence $n_0 = 2$ and the minmax condensation of $\mathfrak{C}$ is *Can*([3/16, 1/16], [1/8, 1/16, 1/16]). A canon class always has the same resultant as its minmax condensation; hence *Res* $\mathfrak{C}$ = [3/16, 1/16] + [1/8, 1/16, 1/16] = [1/16].

3. Complementary canons

One usually says that two voices are complementary if no attack of one voice is simultaneous of any attack of the other. Within the rhythmic model of Section 1, the fact that two voices delivering the periodic rhythms R_1, R_2 respectively are complementary is expressed by the equality $R_1 \cap R_2 = \emptyset$. When applied to canons, these considerations

lead to the following definition:

DEFINITION 3.1. *A canon* $\{R_1, \ldots, R_l\}$ *is called complementary if* $R_i \cap R_j = \emptyset$ *for* $i \neq j$.

The next proposition shows the close connection between complementarity of canons and intervallic disjointness of rhythmic classes.

PROPOSITION 3.1. *For any canon* $\mathfrak{C}$ *the following conditions are equivalent:*

i) $\mathfrak{C}$ *is complementary;*

ii) $Grd\ \mathfrak{C} \perp Met\ \mathfrak{C}$;

iii) Grd $\mathfrak{C}$ *is intervallically disjoint from some metric class admitted by* $\mathfrak{C}$;

iv) Grd $\mathfrak{C}$ *is intervallically disjoint from all metric classes admitted by* $\mathfrak{C}$.

Conversely, given any intervallically disjoint rhythmic classes R and S, the canons in the canon class $Can(R, S)$ are complementary and admit S as a metric class of order $k = (Per\ R \vee Per\ S) / Per\ R$.

We describe now a procedure of "tilling" a complementary canon based on a simple device we have called *elementary derivation* (see next section for an example). By definition, the latter means any of the following transformations applied to a pair (R, S) of intervallically disjoint rhythmic classes:

- replacing R by any $R' \in Rhyt$ satisfying the relations $R' + [Per\ R \vee Per\ S] = R$ and $R' \perp [Per\ R \vee Per\ S]$;
- replacing S by any $S' \in Rhyt$ satisfying the relations $S' + [Per\ R \vee Per\ S] = S$ and $S' \perp [Per\ R \vee Per\ S]$.

The classes in a pair obtained by an application of an elementary derivation are still intervallically disjoint. Moreover, the composition of the two classes in the pair does not change under elementary derivation. Hence, by applying successive elementary derivations to the pair $(Grd\ \mathfrak{C}, Met\ \mathfrak{C})$ where $\mathfrak{C}$ is a complementary canon, the composer has the opportunity to till the canon $\mathfrak{C}$ by enlarging both its temporal dimension (the period of its ground class) as well as its spatial dimension (the number of the voices) *without changing its resultant class*. It turns in fact that there is a stronger invariant with respect to these transformations, which allows one to assert that, despite the apparently increasing complexity of the successively derived canons, all of them continue to keep (although in a hidden way) the simplicity of the starting pattern $(Grd\ \mathfrak{C}, Met\ \mathfrak{C})$. This invariant is provided by the minmax condensation.

PROPOSITION 3.2. *Let R, $S \in$ Rhyt be such that $R \perp S$ and let (R', S') be obtained from (R, S) by elementary derivations. Then the minmax condensations of Can(R, S) and of Can(R', S') coincide.*

PROPOSITION 3.3. *Let $\mathfrak{C}$ be a complementary canon and let Can(R_0, S_0) be the minmax condensation of $\mathfrak{C}$ represented via a normal pair. Then the pair (Grd $\mathfrak{C}$, Met$\mathfrak{C}$) is obtained from the normal pair (R_0, S_0) by applying a finite number of successive elementary derivations.*

Thus, the conclusion of this section is that the true complexity of complementary canons is carried by those having maximal category. Once we know how to construct the latter, we have all the other by elementary derivations.

4. The structure of regular complementary canons

DEFINITION 4.1. *A regular complementary canon is a complementary canon whose resultant class is regular.*

We have seen in the preceding section that the notion of a complementary canon was related to the notion of intervallically disjoint rhythmic classes. Following the same idea, regular complementary canons are related to the notion of supplementary rhythmic classes to be introduced below.

DEFINITION 4.2. *Two rhythmic classes R, S are called supplementary if $R \perp S$ and $R + S$ is regular.*

PROPOSITION 4.1. *For any canon $\mathfrak{C}$ the following conditions are equivalent:*

i) $\mathfrak{C}$ is regular and complementary;

ii) Grd $\mathfrak{C}$ is supplementary to Met $\mathfrak{C}$;

iii) Grd $\mathfrak{C}$ is supplementary to some metric class admitted by $\mathfrak{C}$;

iv) Grd $\mathfrak{C}$ is supplementary to any metric class admitted by $\mathfrak{C}$.

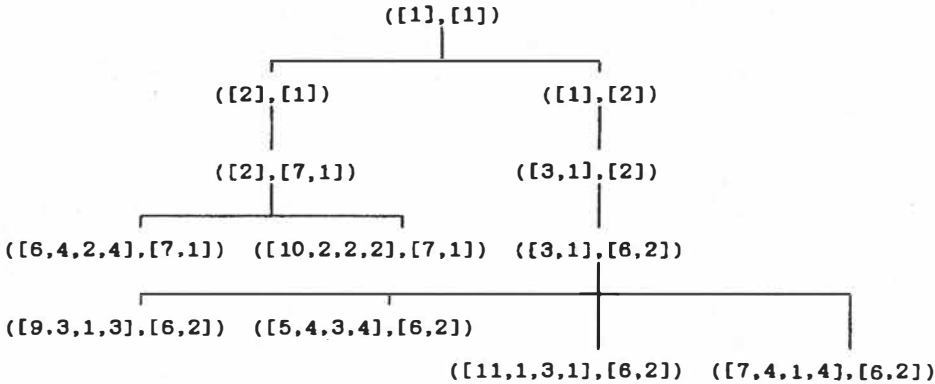
Since elementary derivation does not change the resultant class, it follows that we get a supplementary pair whenever we apply an elementary derivation to any supplementary pair of rhythmic classes. The simplest conceivable such pairs are those of the form $([t], [t])$ for some $t \in \mathbb{Q}_+$, which will be called *elementary*. A canon whose class equals Can$([t], [t])$ will also be called elementary.

DEFINITION 4.3. *A regular complementary canon is said to be constructible by elementary derivations if it can be represented as Can(R, S) where the pair (R, S) is obtained by applied a finite number of*

successive elementary derivations to some elementary pair.

Taking into account the results of Section 3, we see that a canon is constructible by elementary derivations iff its minmax condensation is the class of an elementary pair.

Here are some examples of successive elementary derivations which start from the elementary pair ([1],[1]).



It is now the time to make clear the relation between supplementary sets and regular complementary canons we have spoken about in the Introduction.

DEFINITION 4.4. *Two subsets* M, N *of* $\mathbb{Z}_n$ *are called supplementary if every* $x \in \mathbb{Z}_n$ *can be uniquely written as* $y + z$ *with* $y \in$ M *and* $z \in$ N.

Two translation classes of $\mathbb{Z}_n$ *are called supplementary if they are the classes of two supplementary subsets of* $\mathbb{Z}_n$.

PROPOSITION 4.2. *Let* R, $S \in$ *Rhyt be given, let* a = Div $R \wedge$ Div S *and let* b = Per $R \vee$ Per S. *Then* R *and* S *are supplementary rhythmic classes iff* $H_{a,b}(R)$ *and* $H_{a,b}(S)$ *are supplementary translation classes of* $\mathbb{Z}_n$, *where* $n = b/a$.

The relation we have just spoken about is thus clarified by the above result together with Proposition 4.1.

Thus, we see that seeking for regular complementary canons amounts to seeking for pairs of supplementary classes in $T(\mathbb{Z}_n)$ for n an arbitrary integer ≥ 1. Given a regular complementary canon $\mathfrak{C}$, set a = Div *Res* $\mathfrak{C}$ and b = Per *Grd* $\mathfrak{C}$. $\mathfrak{C}$ is of maximal category iff $H_{a,b}$(*Met* $\mathfrak{C}$) is nonperiodic; a translation class M is said *periodic* if it is the class of some periodic subset M $\subset \mathbb{Z}_n$, the latter meaning that t + M = M for some $t \in \mathbb{Z}_n \setminus \{0\}$. Since $H_{a,b}$(*Met* $\mathfrak{C}$) is always nonperiodic (because of b = Per *Grd* $\mathfrak{C}$), we see that seeking for regular complementary

canons of maximal category amounts to seeking for pairs of supplementary nonperiodic classes in $T(\mathbb{Z}_n)$.

Observe that n can take here arbitrary large values, a fact which definitely distinguishes the rhythmic context from its pitch-class analog described in the Introduction, where n was taking the only value 12. If in the pitch-class context we could still hope to be able to check the truth of some result by inspecting a finite (although lengthy) list of possibilities, such a verification is here no longer possible and the resort to the mathematical methods developed in [6] has become a strict necessity.

We know, according to Section 3, that the regular complementary canons which are constructible by elementary derivations are the regular complementary canons with the simplest conceivable structure. The next two theorems, which represent the main results in [6], show that such a constructibility is closely related to property of the numerical invariants described invariants described in Section 1 to be "not too complicated" numbers. They also show that the constructibility is closed related to the maximality of the category. In particular, they assert that *there exist nonelementary regular complementary canons of maximal category,* which is not an obvious fact insofar as the regular complementary canons of not too large size, which occur in most common situations, must obey those theorems, which strictly forbid their category to be maximal.

Before stating the theorems we introduce certain sets of positive integers (by a prime we mean any prime number ≥ 2):

$$N_0 = \{p^k \mid p \text{ prime}, k \geq 0\};$$
$$N_1 = \{p^k q \mid p, q \text{ distinct primes}, k \geq 1\};$$
$$N_2 = \{p^2 q^2 \mid p, q \text{ distinct primes}\};$$
$$N_3 = \{p^k qr \mid p, q, r \text{ distinct primes}, k \in \{1, 2\}\};$$
$$N_4 = \{pqrs \mid p, q, r, s \text{ distinct primes}\};$$
$$N = \bigcup_{i=0}^{4} N_i .$$

THEOREM 4.1. *For every integer $m \geq 1$ the following conditions are equivalent:*

i) $m \in N_0$;

ii) Every nonelementary regular complementary canon whose ground number or whose category equals m is not a canon of maximal category;

iii) Every regular complementary canon whose ground number or whose

category equals m is constructible by elementary derivations.

THEOREM 4.2. *For every integer $n \geq 1$ the following conditions are equivalent:*

i) $n \in \mathbb{N}$;

ii) Every nonelementary regular complementary canon of modulus n is not a canon of maximal category;

iii) Every regular complementary canon of modulus n is constructible by elementary derivations.

COROLLARY. *A regular complementary canon on p^k voices (p prime, $k \geq 1$) is constructible by elementary derivations and it is not a canon of maximal category.*

By the remark following Propositions 4.2, it is seen that Theorems 4.1 and 4.2 are just the rhythmic interpretation of the next two theorems about supplementary sets.

THEOREM 4.3. *For every integer $m \geq 1$ the following conditions are equivalent:*

i) $m \in \mathbb{N}_0$;

ii) For every integer $n \geq 2$ and every pair M, N of supplementary subsets of $\mathbb{Z}_n$ such that the number of elements in M equals m, it is true that at least one of the subsets M, N is periodic.

THEOREM 4.4. *For every integer $n \geq 2$ the following conditions are equivalent:*

i) $n \in \mathbb{N}$;

ii) In every pair of supplementary subsets of $\mathbb{Z}_n$, at least one of the subsets is periodic.

The proof of the implications *ii)* $\Rightarrow$ *i)* in the above theorems are an immediate consequence of the following proposition:

PROPOSITION 4.3. *Suppose that $n = p_1p_2n_1n_2n_3$ with p_1, p_2 primes, $n_i \geq 2$ for $1 \leq i \leq 3$ and p_1n_1 relatively prime to p_2n_2. Then there are nonperiodic supplementary subsets M, N of $\mathbb{Z}_n$ such that the number of elements in M, N are n_1n_2 and $p_1p_2n_3$ respectively.*

The proof of the above result is based on an algorithm which effectively constructs the sets in question. Since it is that algorithm which leads to the construction of regular complementary canons of maximal category, we would like to present it shortly. To this purpose, we first recall some algebraic concepts. Let G be a commutative group and H be a subgroup of G. The subsets of G in the translation class of G (hence, the sets of the form $x + H$ as x ranges over G) are called the *cosets of* G *modulo* H. Such cosets are either equal or dis-

joint, hence they form a partition of G. By *a set of representants of* G *modulo* H is meant any subset of G which meets every coset modulo H at one and only one element. We also recall that for every divisor d of n there is one and only one subgroup of $\mathbb{Z}_n$ having n/d elements; it is denoted by $d\mathbb{Z}_n$.

The description of the algorithm follows; we are using the notations of Proposition 4.3.

- For $i = 1, 2$ choose any nonperiodic subset M_i of representants of $(n/p_i n_i)\mathbb{Z}_n$ modulo its subgroup $(n/p_i)\mathbb{Z}_n$;
- Set $M = M_1 + M_2$;
- For $i = 1, 2$ choose any x_i in $(n/p_i n_i)\mathbb{Z}_n \setminus (n/p_i)\mathbb{Z}_n$ (set-theoretic difference) and set

$$N_1 = (n/p_1)\mathbb{Z}_n + (\ ((n/p_2)\mathbb{Z}_n \setminus \{0\}) \cup \{x_1\}\ ),$$

$$N_2 = (n/p_2)\mathbb{Z}_n + (\ ((n/p_1)\mathbb{Z}_n \setminus \{0\}) \cup \{x_2\}\ );$$

- Choose any set R of representants of $\mathbb{Z}_n$ modulo $n_3\mathbb{Z}_n$ and set $N = N_1 \cup (N_2 + (R \setminus n_3\mathbb{Z}_n))$.

Once we have found two supplementary nonperiodic subsets $M, N \subset \mathbb{Z}_n$, we may take any $a \in \mathbb{Q}_+$ and set $R = H^{-1}_{a,na}([M])$, $S = H^{-1}_{a,na}([N])$. Then $Can(R, S)$ is the class of a regular complementary canon of maximal category.

It follows from the above theorems that at least six voices are required to construct a nonelementary regular complementary canon of maximal category and that the least modulus of such a canon is 72. Thus, to construct one of the "simplest" regular complementary canon of maximal category we need to find two nonperiodic supplementary subsets of $\mathbb{Z}_{72}$. We give their construction in order to illustrate an application of the above described algorithm.

Take $p_1 = 2$, $p_2 = 3$, $n_1 = 2$, $n_2 = 3$, $n_3 = 2$, $n = 2$ so that $p_1 n_1$ is relatively prime to $p_2 n_2$ and $n = p_1 p_2 n_1 n_2 n_3 = 72$. The subgroups of $\mathbb{Z}_{72}$ to be needed in the following are:

$36\mathbb{Z}_{72} = \{0, 36\}$,

$24\mathbb{Z}_{72} = \{0, 24, 48\}$,

$18\mathbb{Z}_{72} = \{0, 18, 36, 54\}$,

$8\mathbb{Z}_{72} = \{0, 8, 16, 24, 32, 40, 48, 56, 64\}$

and $2\mathbb{Z}_{72}$, the subgroup with 36 elements.

Choose a nonperiodic set M_1 of representants of $18\mathbb{Z}_{72}$ modulo its subgroup $36\mathbb{Z}_{72}$:

$$M_1 = \{0, 18\}.$$

Choose a nonperiodic set M_2 of representants of $8\mathbb{Z}_{72}$ modulo its subgroup $24\mathbb{Z}_{72}$:

$$M_2 = \{0, 32, 40\}.$$

Form M:

$$M = M_1 + M_2 = \{0, 18, 32, 40, 50, 58\}.$$

Choose x_1 in $18\mathbb{Z}_{72} \setminus 36\mathbb{Z}_{72}$: $x_1 = 18$.

Choose x_2 in $8\mathbb{Z}_{72} \setminus 24\mathbb{Z}_{72}$: $x_2 = 8$.

The set $R \setminus 2\mathbb{Z}_{72}$ is here a set consisting of a single element y chosen from $\mathbb{Z}_{72} \setminus 2\mathbb{Z}_{72}$; we take $y = 9$.

Form

$$N_1 = \{0, 36\} + \{18, 24, 48\} = \{12, 18, 24, 48, 54, 60\}$$

and

$$N_2 = \{0, 24, 48\} + \{8, 36\} = \{8, 12, 32, 36, 56, 60\}.$$

Finally form N:

$$N = N_1 \cup (9 + N_1)$$

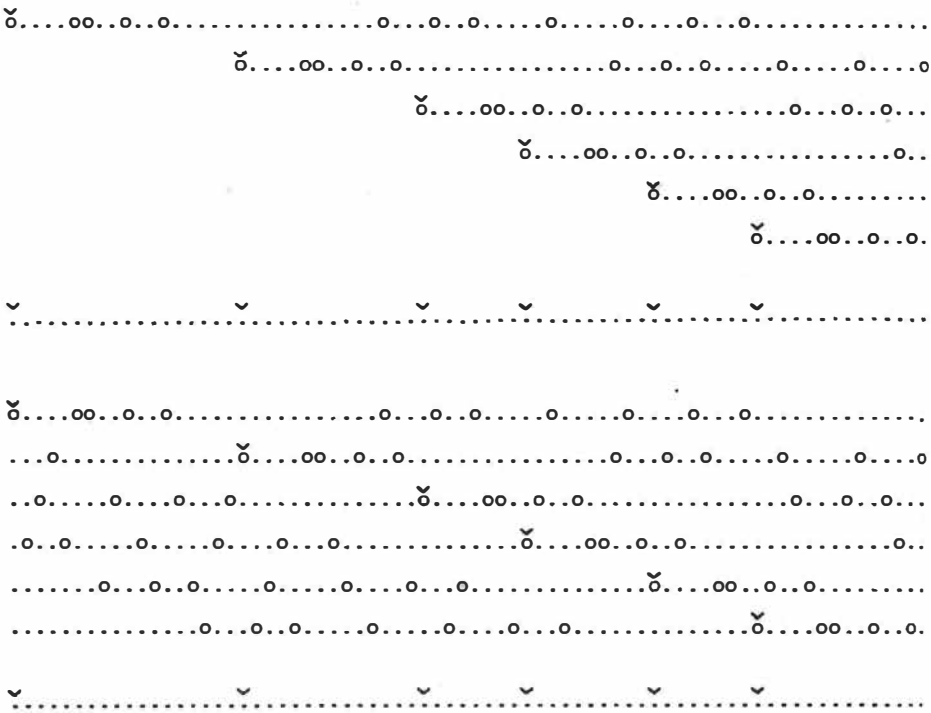
$$= \{12, 17, 18, 21, 24, 41, 45, 48, 54, 60, 65, 69\}.$$

The rhythmic classes corresponding to [N] and [M] via $H_{1,72}$ are

$$R = H_{1,72}^{-1}([N]) = [5, 1, 3, 3, 17, 4, 3, 6, 6, 5, 4, 15],$$

$$S = H_{1,72}^{-1}([M]) = [18, 14, 8, 10, 8, 14].$$

A regular complementary canon of maximal category whose class equals $Can(R, S)$ is presented below.



The proof of the implications *i*) ⇒ *ii*) in the Theorems 4.3 - 4.4 are more elaborate and rely on the applications of the methods of Discrete Harmonic Analysis. The principle is the following: the fact that M and N are supplementary can be expressed in terms of convolution on the group $\mathbb{Z}_n$ as $\chi_M * \chi_N = C$, where χ_M and χ_N denote the characteristic functions of M and N respectively and C denotes the function identically one on $\mathbb{Z}_n$. By taking Fourier transforms we obtain $\chi_M\hat{}(\omega)\chi_N\hat{}(\omega) = 0$ for every n-th root of unity $\omega \neq 1$. As $\chi_M\hat{}(\omega)$ and $\chi_N\hat{}(\omega)$ are polynomials in ω with rational coefficients we take advantage of the irreducibility of the n-th cyclotomic polynomial in order to obtain that the equality $\chi_M\hat{}(\omega)\chi_N\hat{}(\omega) = 0$ implies stronger conditions on χ_M and χ_N which can be expressed in terms of convolution and have to be carefully analyzed. The reader is referred to [6] for the details.

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